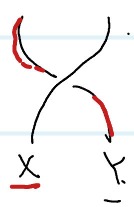
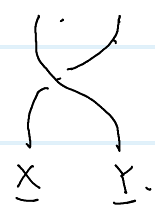
Def: $\mathcal{C}$ monoidal category.

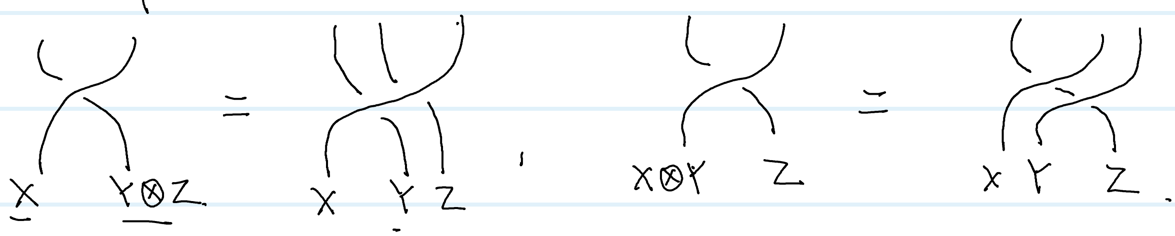
Braiding is a family of isomorphisms

$$\tau = \{\tau_{X,Y} : X \otimes Y \to Y \otimes X\}_{X,Y \in Ob(\mathcal{C})}.$$

Denote

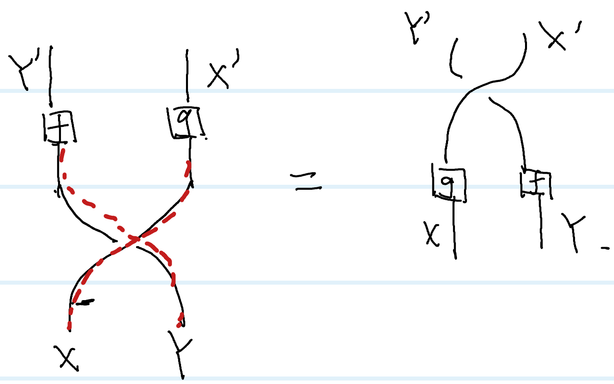
$\tau_{X,Y} =$  , $\tau^{-1}_{Y,X} =$ 

s.t (1) $\otimes$-multiplicative.



(2) naturality

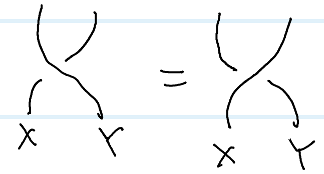
$\forall f: Y \to Y'$, $g: X \to X'$.



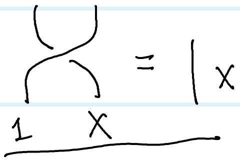
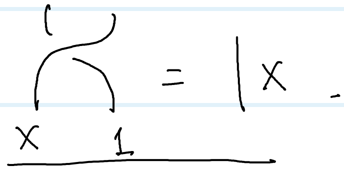
A braided category is monoidal category endowed with a braiding. $(\mathcal{C}, \tau)$

A braided functor between braid categories $(\mathcal{C}, \tau)$ and $(\mathcal{C}', \tau')$ is a monoidal functor $F: \mathcal{C} \to \mathcal{C}'$. s.t

$$F_2(X,Y)\, \tau'_{F(X), F(Y)} = F(\tau_{X,Y})\, F_2(X,Y), \quad \forall X, Y \in Ob(\mathcal{C})$$

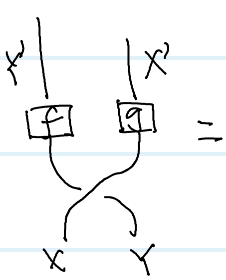
A braid category is symmetric, if  , $\forall X, Y \in ob(\mathcal{C})$.

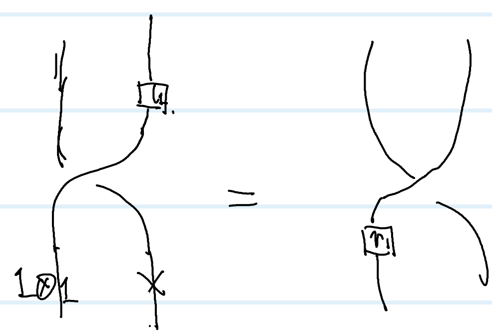
$$\tau^{-1}_{Y,X} = \tau_{X,Y}$$

Lem:  ,  .

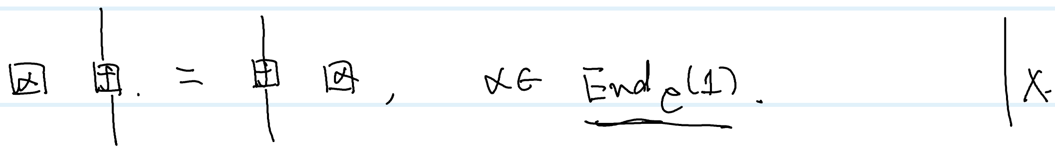
Pf; $\underline{\tau_{1,X} = id_X}$. $\underline{r_X \tau_{1,X} = l_X}$. $\underline{id_1} \otimes$

$$\begin{tikzcd}
 & (1\otimes 1)\otimes X \arrow[r, "\tau_{1\otimes 1, X}"] \arrow[d, "r_1 \otimes id_X"] \arrow[dr, phantom, "\circlearrowleft"] & X \otimes (1 \otimes 1) \arrow[d, "id_X \otimes l_1 = id_X \otimes r_1"] \arrow[dr, "a^{-1}_{X,1,1}"] & \\
1 \otimes (1 \otimes X) \arrow[ur, "a^{-1}_{1,1,X}"] \arrow[r, "id_1 \otimes l_1"] \arrow[dr, "id_1 \otimes \tau_{1,X}"'] & 1 \otimes X \arrow[r, "\tau_{1,X}"] & X \otimes 1 & (X \otimes 1)\otimes 1 \arrow[l, "r_{X\otimes 1}"'] \\
 & 1 \otimes (X \otimes 1) \arrow[u, "id_1 \otimes r_X"] \arrow[r, "a^{-1}_{1,X,1}"'] & (1 \otimes X)\otimes 1 \arrow[ul, "r_{1\otimes X}"] \arrow[ur, "\tau_{1,X} \otimes id_1"'] &
\end{tikzcd}$$



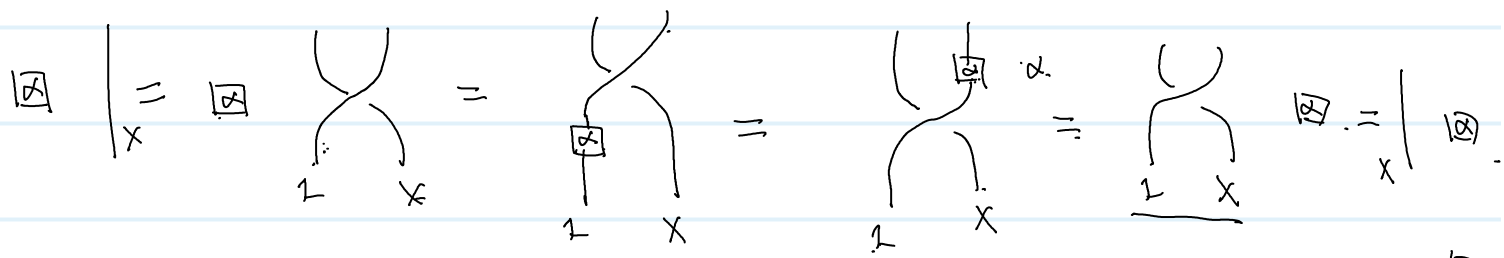


Lem: braid category is pure.



$\alpha \in \underline{\mathrm{End}_{\mathcal{C}}(1)}$.

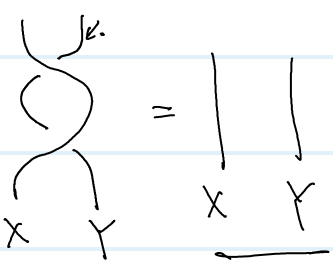
pf:



$\square$

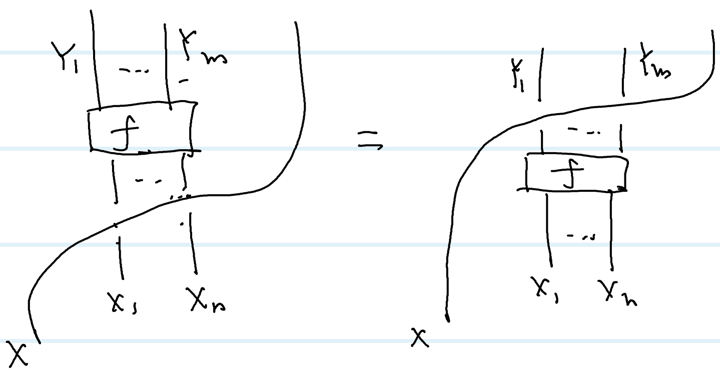
Lem: $\mathcal{C}$ braided pivotal category. Then

(1)

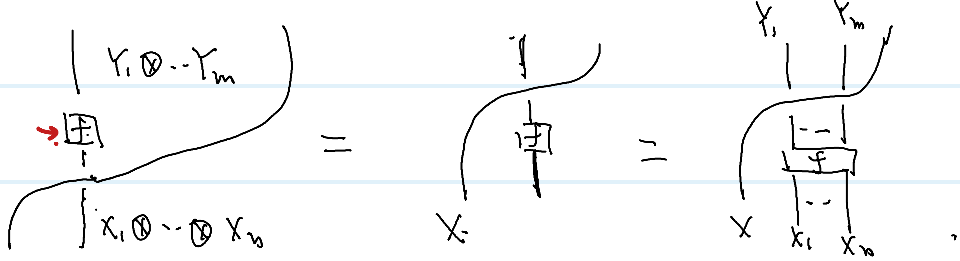


, $\tau_{X,Y}^{-1} \cdot \tau_{X,Y} = \mathrm{id}_{X\otimes Y}$.

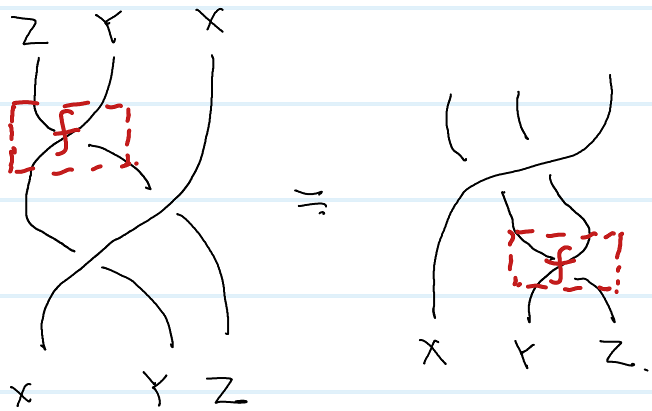
(2)



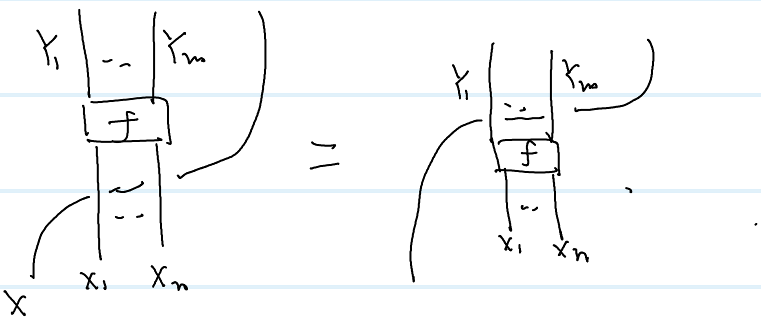
The left.



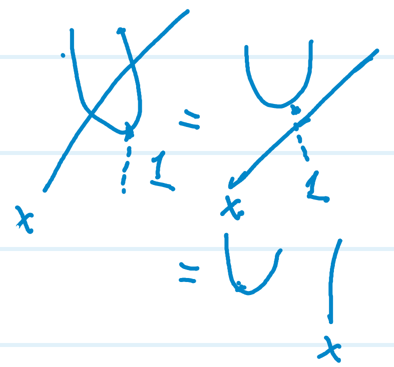
(3)



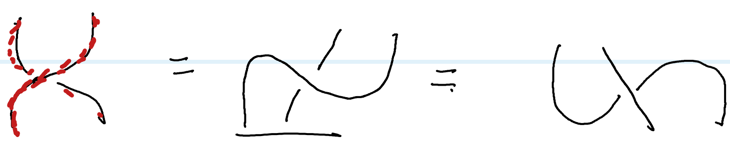
(4)



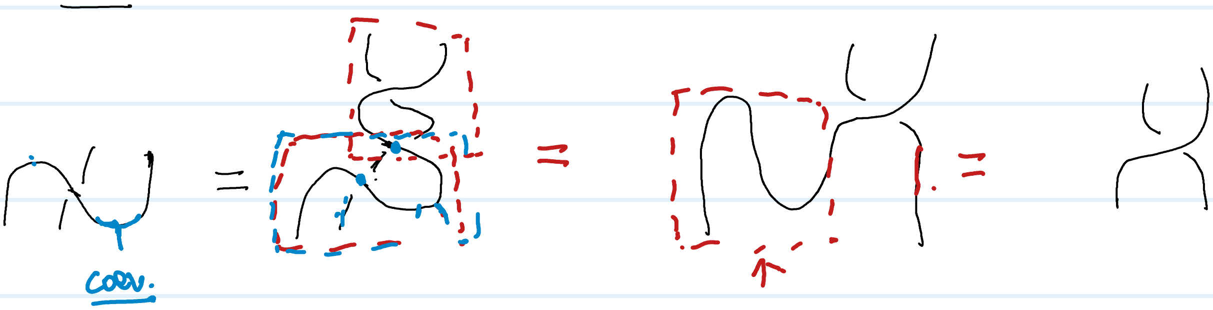
Lem: $\mathcal{C}$ braided pivotal category.



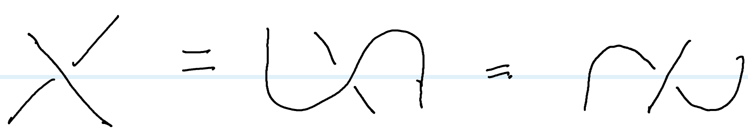
(1).



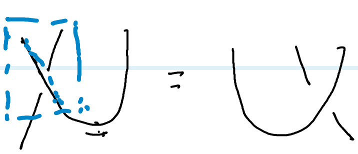
Pf:



(2)

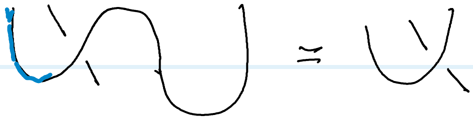


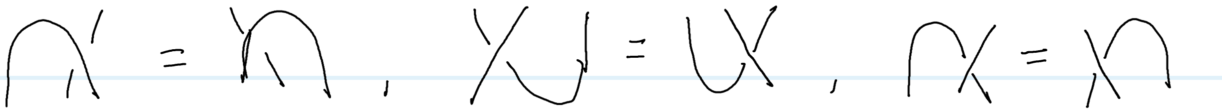
(3)



coev.

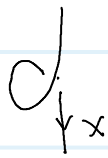
pf



(14) 

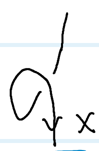
$\mathcal{C}$ braided pivotal category.

Def: left twist of $X \in ob(\mathcal{C})$.

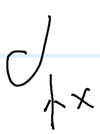
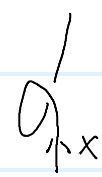
$\theta^l_X =$  $: X \to X$.

right twist $\theta^r_X =$  $: X \longrightarrow X$.

Lem: The twists. θ^l and θ^r are natural isomorphisms $(1_\mathcal{C} \to 1_\mathcal{C})$.

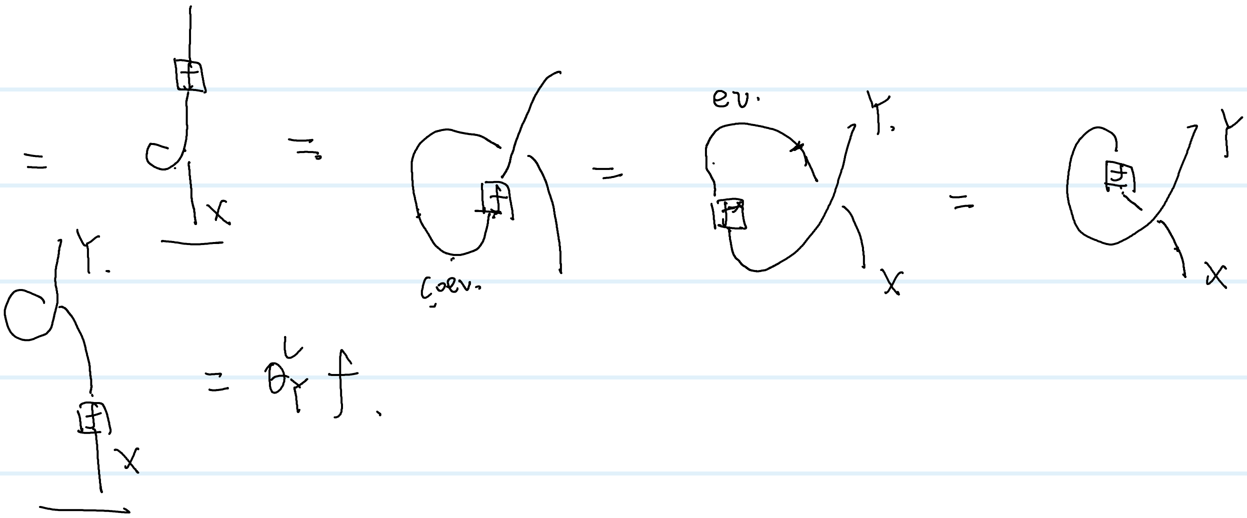
$(\theta^l_X)^{-1} =$  , $(\theta^r_X)^{-1} =$  .

Moreover, $(\theta^r_X)^* = \theta^l_{X^*}$, $(\theta^l_X)^* = \theta^r_{X^*}$. and.

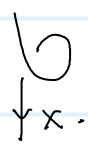
$\theta^l_{X^*} =$  , $\theta^r_{X^*} =$  , $(\theta^l_{X^*})^{-1} =$  , $(\theta^r_{X^*})^{-1} =$  .

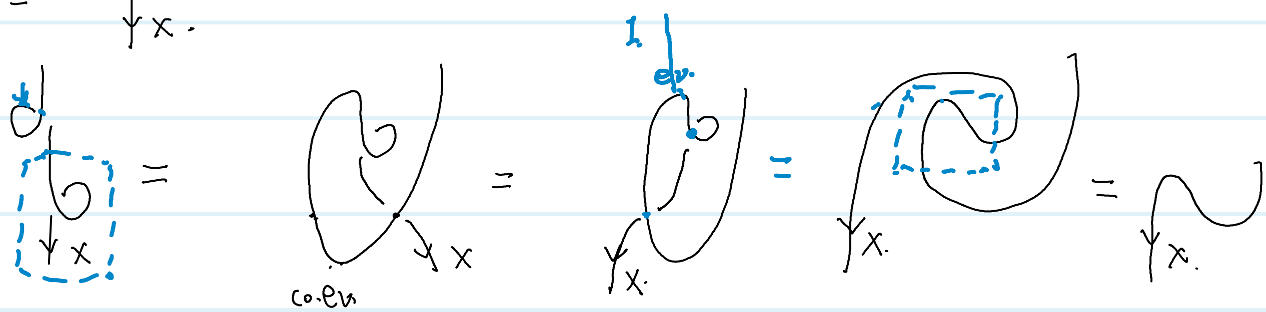
pf: naturality of θ^l.

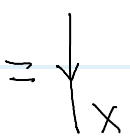
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\theta^l_X"'] \arrow[dr, phantom, "?"] & Y \arrow[d, "\theta^l_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$f\theta^l_X =$ 

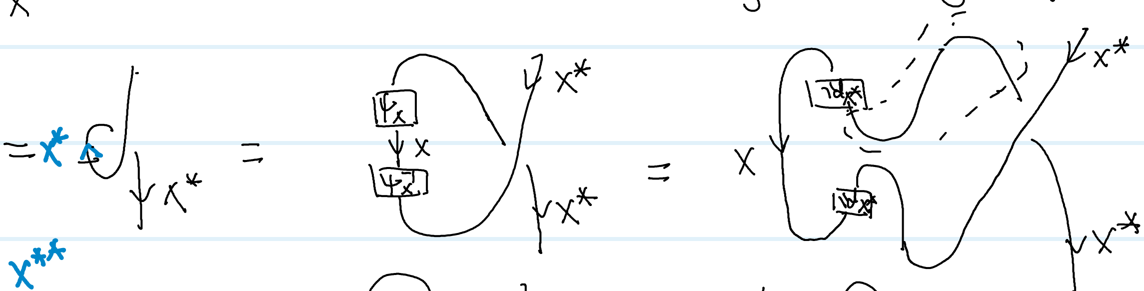
$= \theta^l_Y f$.

Set $v^l_X =$ 

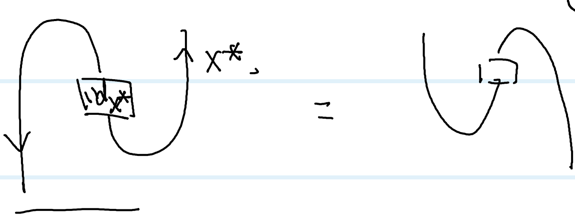
$\theta^l_X v^l_X =$ 

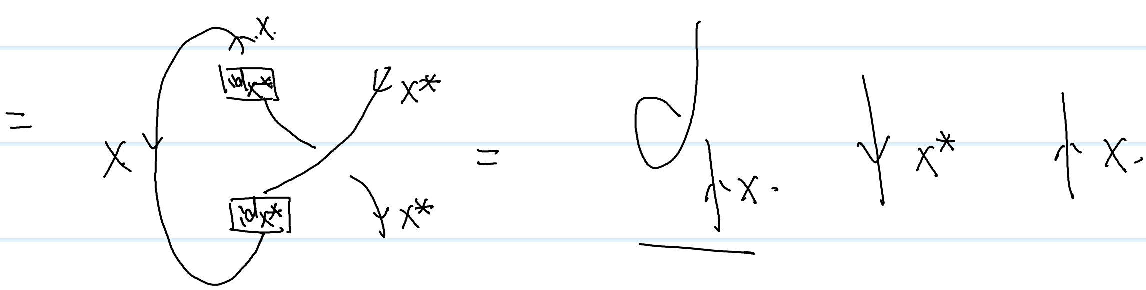
$=$ 

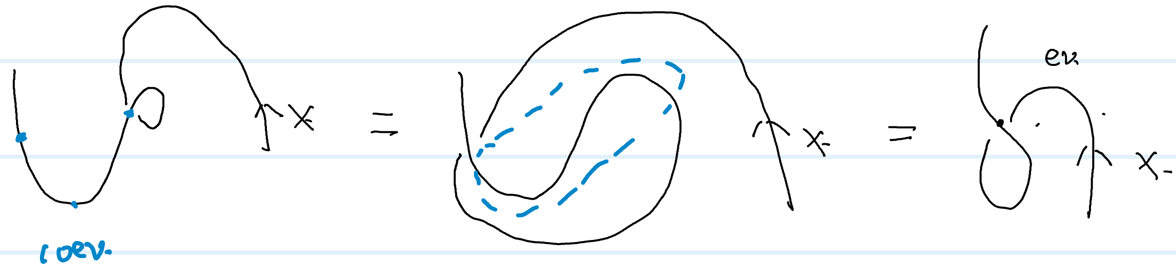
, $v^l_X \theta^l_X = \theta^l_X v^l_X = id_X$. by naturality of θ^l.

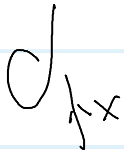
$\theta^l_{X^*} =$ 

$\psi_X : X \to X^{**}$.





$(\theta^r_X)^* =$ 

$=$ 

$= \theta^l_{X^*}$.

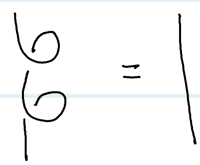
$tr_r(\theta^l_X) =$ 

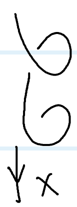
$= tr_l(\theta^r_X)$

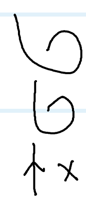
Def: A braided pivotal category is ribbon, if $\theta^l = \theta^r$.

Twist of $\mathcal{C}$ is $\theta = \theta^l = \theta^r$.

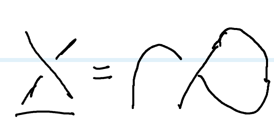
$(\theta_X)^* = \theta_{X^*}$.

Lem: $\mathcal{C}$ ribbon category, then 

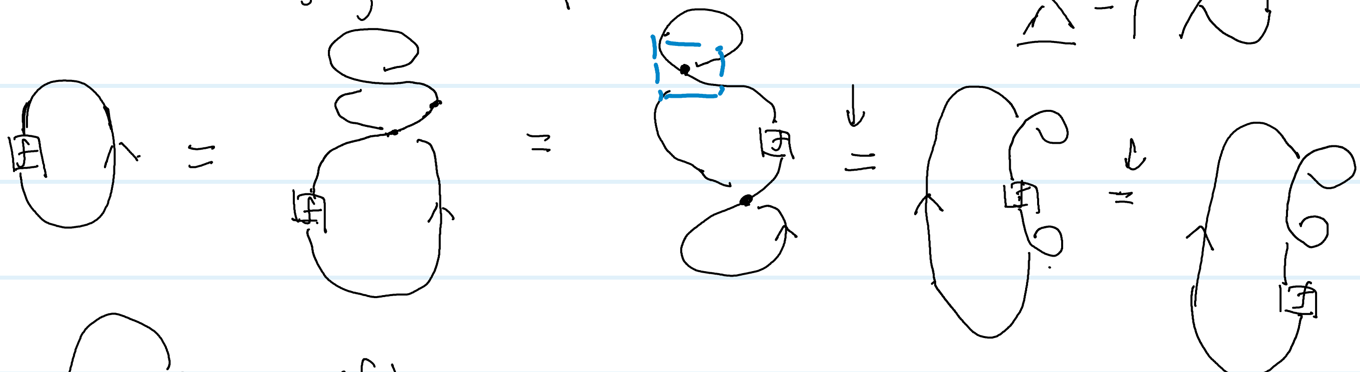
Pf: 

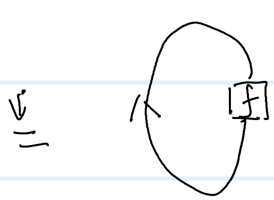
$= \theta^r_X (\theta^l_X)^{-1} = id_X$, 

$= \underline{\theta^r_{X^*}} (\theta^l_{X^*})^{-1} = id_{X^*}$. □

Cor: Ribbon category is spherical. 

Pf:

$tr_r(f) =$ 

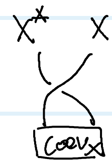


$= tr_l(f)$.

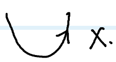
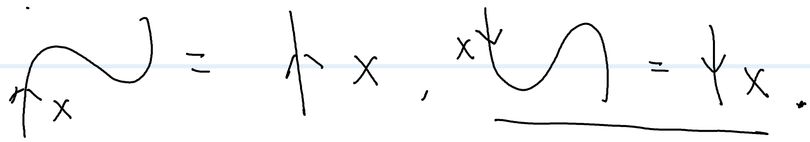
lem: $(\mathcal{C}, \tau)$ symmetric category, a left duality $\{({}^\vee X, ev_X)\}_{X\in Ob(\mathcal{C})}$

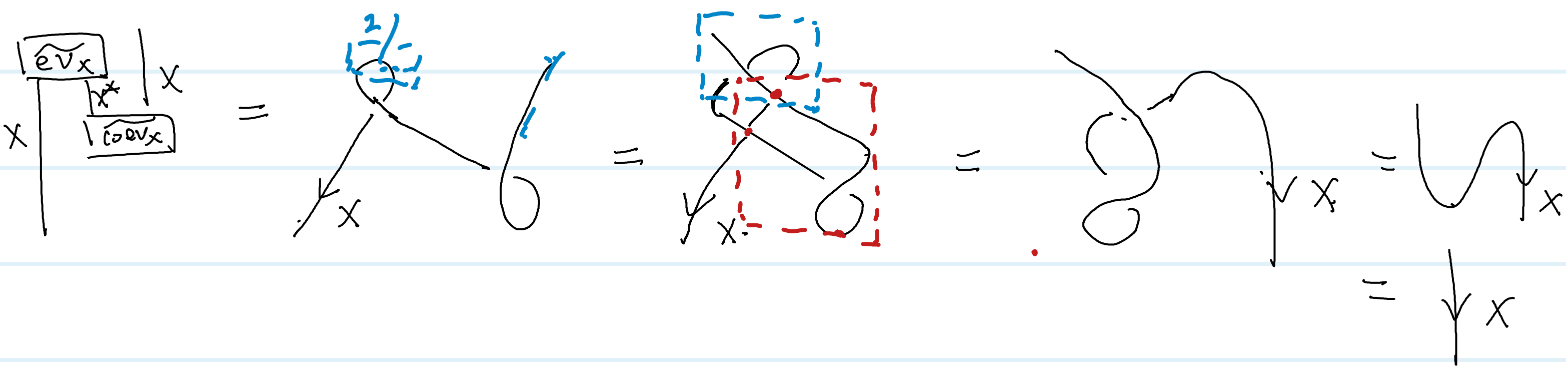
$ev_X : {}^\vee X \otimes X \to 1$, its inverse $coev_X : 1 \to X \otimes {}^\vee X$.

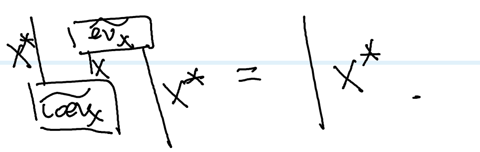
Set $X^* = {}^\vee X$.

$\widetilde{ev}_X =$  $: X \otimes X^* \to 1$, $\widetilde{coev}_X =$  $: 1 \to X^* \otimes X$.

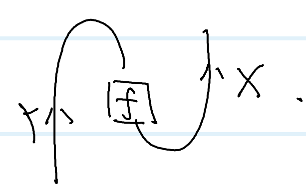
(1) $\mathcal{C}$ is pivotal category.

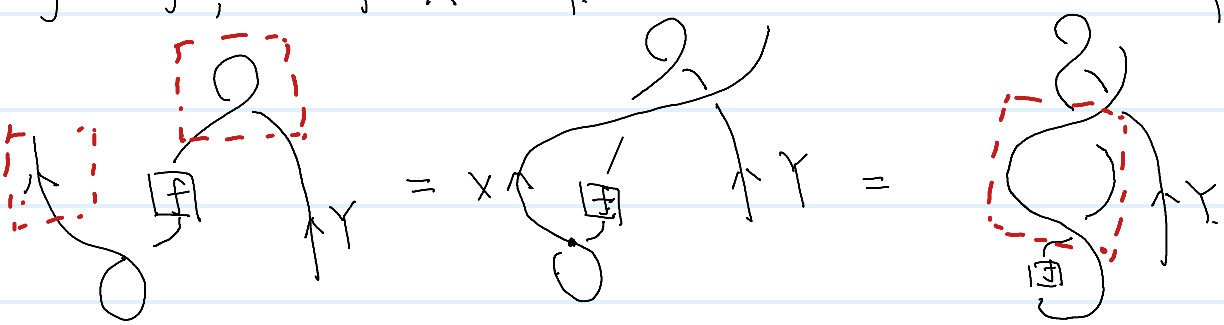
$ev_X =$ , $coev_X =$ , 

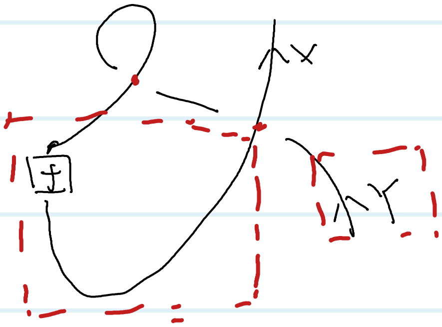
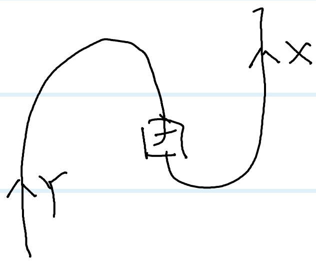


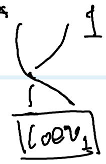
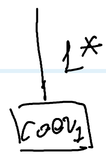


${}^\vee f = f^\vee$, $f: X \to Y$.

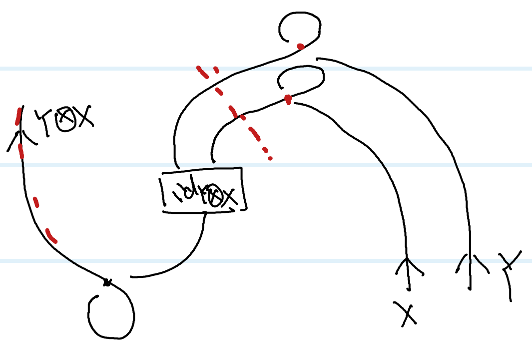


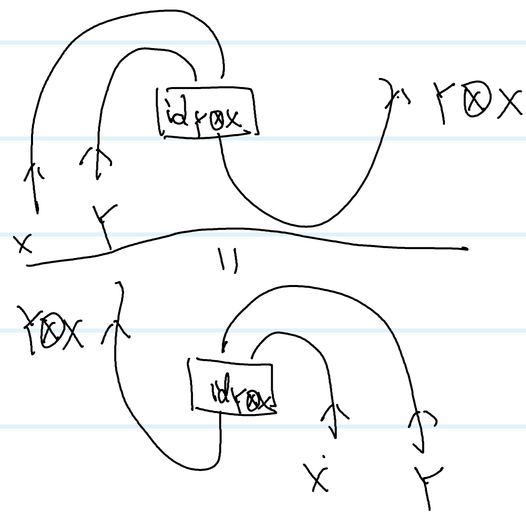


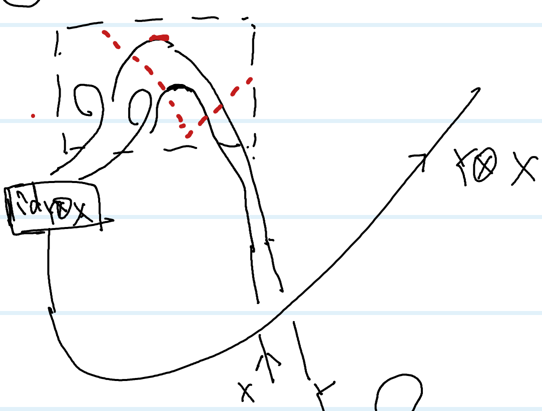
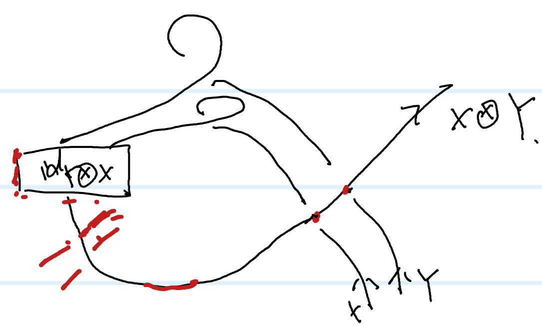
=  = 

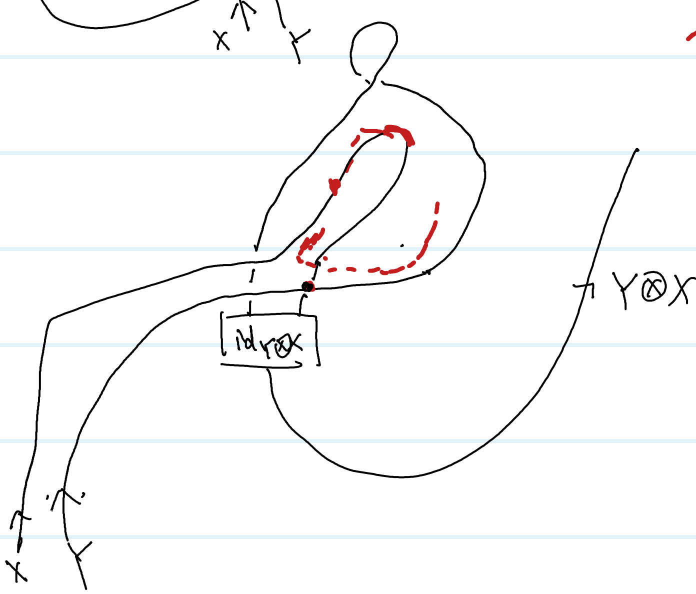
$({}^\vee ?)_0 = (?^\vee)_0$, $\widetilde{coev}_1 =$  =  $= coev_1$

$({}^\vee ?)_2 = (?^\vee)_2$. $X^* \otimes Y^* \longrightarrow (Y \otimes X)^*$.



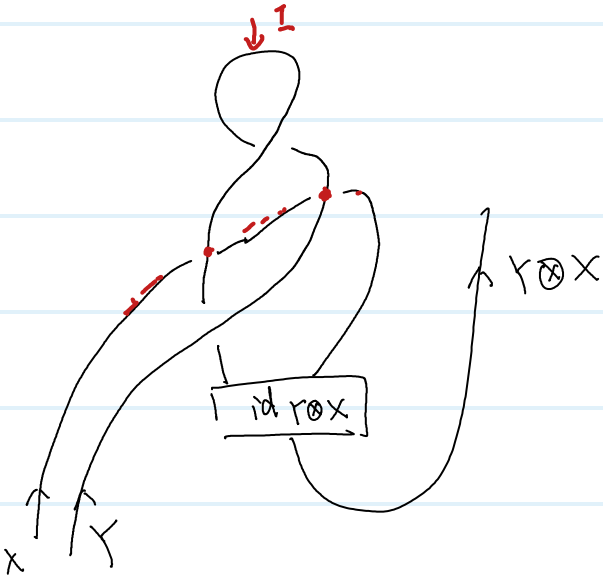


=  = 

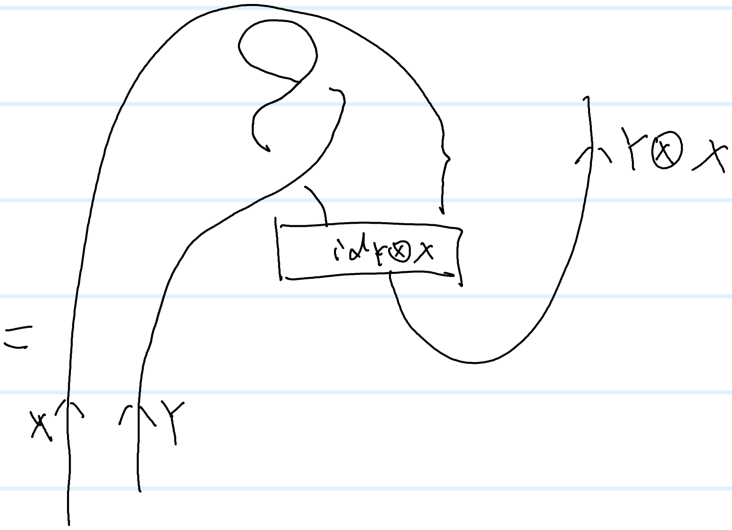
= 

$\otimes$ - multi;

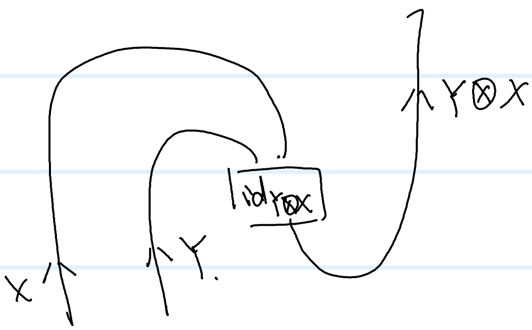
$=$



$=$



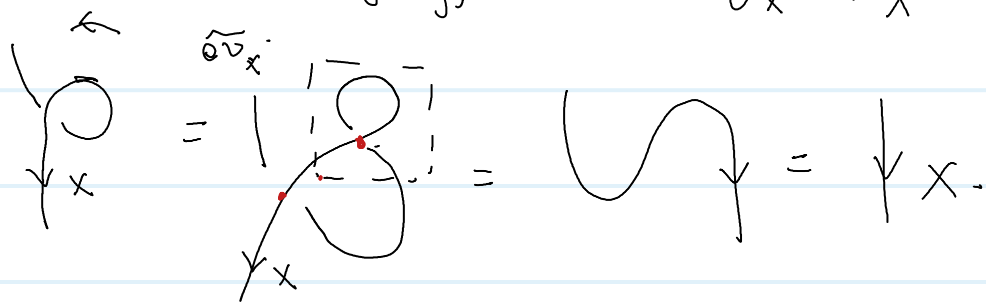
$=$



$\square$

(2) $(\mathcal{C}, \tau)$ is a ribbon category, twist $\theta_X = id_X$, $\forall X \in Ob(\mathcal{C})$.

Pf: $\theta^r_X =$



$\theta^l_X = id_X$.

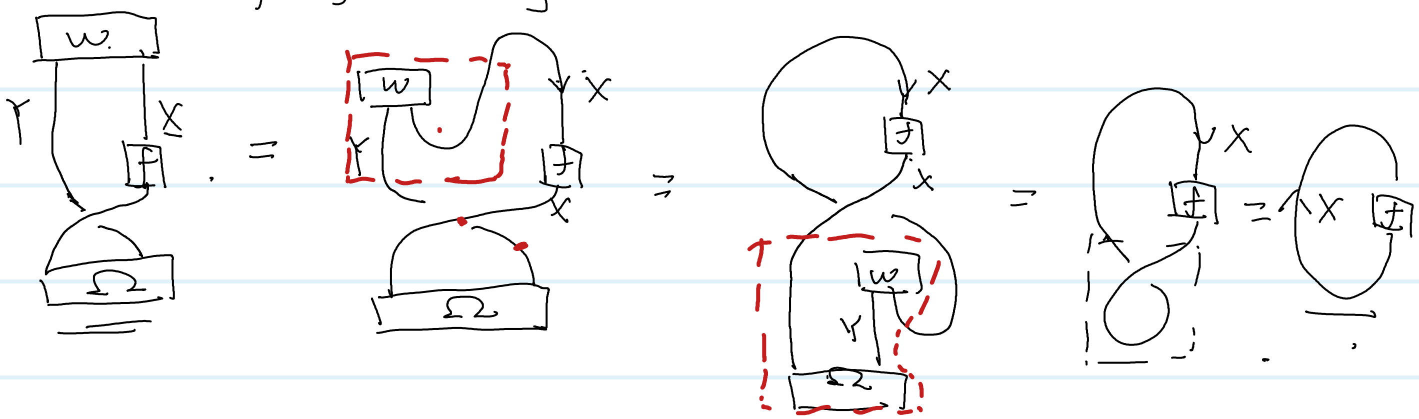
(3). The traces of endomorphisms in $\mathcal{C}$ do not depend on the choice of left duality in $\mathcal{C}$.

Pf: $f: X \to X$. $\check{X}$.

let $w: Y \otimes X \to 1$ another left dual.

its inverse $\Omega: 1 \to X \otimes Y$.

Def trace of f using w and Ω.



$\square$