$\mathbb{K}$: a commutative ring with unity, $\mathrm{Mod}_{\mathbb{K}}$: $\mathbb{K}$-mod category.

Then $\mathrm{Mod}_{\mathbb{K}}$ is a monoidal category with unit object $\mathbb{K}$ ($\mathbb{K} \otimes_{\mathbb{K}} X \simeq X$).

Pairing in $\mathrm{Mod}_{\mathbb{K}}$.

A pairing $\omega : X \otimes_{\mathbb{K}} Y \to \mathbb{K}$, $X, Y \in \mathrm{ob}(\mathrm{Mod}_{\mathbb{K}})$

If ω has inverse $\Omega : \mathbb{K} \longrightarrow Y \otimes_{\mathbb{K}} X$,

$$*_\omega = \Omega(1_{\mathbb{K}}) \in Y \otimes_{\mathbb{K}} X \text{ is called contraction vector of } \omega.$$

$X \in \mathrm{ob}(\mathrm{Mod}_{\mathbb{K}})$, then $X^* = \mathrm{Hom}_{\mathbb{K}}(X, \mathbb{K}) \in \mathrm{ob}(\mathrm{Mod}_{\mathbb{K}})$

$$\forall\, k \in \mathbb{K},\ f \in X^*,\ (kf)(x) := k\, f(x),\ \forall\, x \in X.$$

Def: $X \in \mathrm{ob}(\mathrm{Mod}_{\mathbb{K}})$ is <u>projective</u> of finite type, if it is a <u>direct summand</u> of a free $\mathbb{K}$-mod of finite rank.

Lem. $\rho : \underline{X} \otimes_{\mathbb{K}} \underline{X^*} \to \mathbb{K}$ in $\mathrm{Mod}_{\mathbb{K}}$, then ρ is non-degenerate
$x \otimes f \longmapsto f(x)$

$\iff X$ is projective of finite type.

Pf: X is projective of finite type.

$\iff \exists\, n \in \mathbb{Z}_{\geq 1}$, $\mathbb{K}$-mod homo $e : \mathbb{K}^n \to X$, $\iota : X \to \mathbb{K}^n$, s.t $\underline{e\iota = \mathrm{id}_X}$.

i.e short exact sequence

$$0 \to \underline{\ker e} \longrightarrow \underline{\mathbb{K}^n} \underset{\iota}{\overset{e}{\rightleftarrows}} \underline{X} \to 0 \quad \text{split.}$$

Denote $\underline{e_i} = e((0, \cdots 0, \underset{\substack{\uparrow \\ i}}{1}, 0, \cdots 0)) \in X$.

$$\underline{\iota_i} = X \xrightarrow{\iota} \mathbb{K}^n \longrightarrow \mathbb{K} \in X^*.$$
$$(k_1, \cdots k_n) \longmapsto k_i.$$

Then $\underline{el = id_X} \Rightarrow x = e(l(x)) = e((l_1(x), \cdots, l_n(x)))$

$$= \sum_{i=1}^{n} l_i(x) e_i.$$

Then X is projective of finite type

$\Leftrightarrow \exists\, n \in \mathbb{Z}_{\geq 1}$, $\{e_i\}_{i=1}^{n} \subseteq X$, $\{l_i\}_{i=1}^{n} \subseteq X^*$ s.t.

$$x = \sum_{i=1}^{n} l_i(x) e_i, \ \forall x \in X.$$

"$\Rightarrow$" X is projective of finite.,

Def $R: \mathbb{K} \to X^* \otimes_{\mathbb{K}} X$ $\mathbb{K}$-linear map.

$$1_{\mathbb{K}} \mapsto \sum_{i=1}^{n} l_i \otimes e_i$$

Then $x \in X$, $f \in X^*$.

$$\underline{(id_{X^*} \otimes \rho)(R \otimes id_{X^*})(f)}(x) = \sum_{i=1}^{n} f(e_i) l_i(x) = f(\underbrace{\sum_{i=1}^{n} l_i(x) e_i}_{x}) = f(x).$$

$$(\rho \otimes id_X)(id_X \otimes R)(x) = \sum_{i=1}^{n} l_i(x) e_i = x.$$

"$\Leftarrow$" ρ has inverse $R: \mathbb{K} \to X^* \otimes X$.

$$1_{\mathbb{K}} \mapsto \sum_{i=1}^{n} l_i \otimes e_i \qquad l_i \in X^*, \ e_i \in X.$$

$\exists\ \{l_i\}_{i=1}^{n} \subseteq X^*$, $\{e_i\}_{i=1}^{n} \subseteq X$.

Then $\forall x \in X$

$$x = (\rho \otimes id_X)(id_X \otimes R)(x) = \sum_{i=1}^{n} l_i(x) e_i. \Rightarrow X \text{ is projective of finite type}$$

$\square$

Lem: $w: X \otimes_{\mathbb{K}} Y \to \mathbb{K}$ in $\mathrm{Mod}_{\mathbb{K}}$. The following three conditions are equivalent.

(1) w is non-degenerate

(2) X is projective of finite type, and

$a: Y \to X^*$ is an isomorphism.

$y \mapsto (x \mapsto \omega(x \otimes y))$

(3) Y is projective of finite type, and

$b: X \to Y^*$ is an isomorphism.

$x \mapsto (y \mapsto \omega(x \otimes y))$.

pf: "(1) $\Leftarrow$ (2)"

X is projective of finite type. then $\rho: X \otimes_{\mathbb{k}} X^* \to \mathbb{k}$ is non-degenerate.

$x \otimes f \mapsto f(x)$.

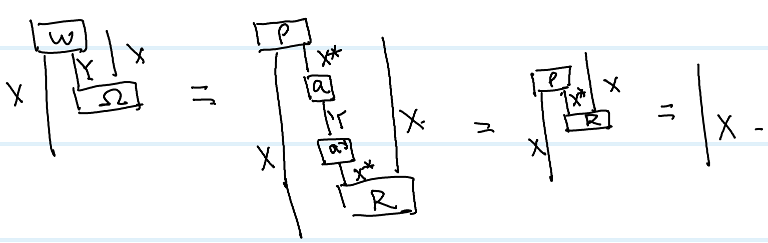
ρ has inverse $R: \mathbb{k} \to X^* \otimes_{\mathbb{k}} X$

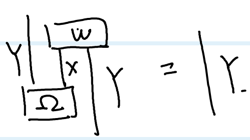
Set $\Omega: \mathbb{k} \xrightarrow{R} X^* \otimes X \xrightarrow{a^{-1} \otimes id_X} Y \otimes X$ $\quad$ $a: Y \to X^*$. $\quad$ $\Omega = (a^{-1} \otimes id_X) R$.

$X \otimes Y \xrightarrow{id_X \otimes a} X \otimes X^* \xrightarrow{\rho} \mathbb{k}$.

Since $\omega(x \otimes_{\mathbb{k}} y) = a(y)(x) = \rho(x \otimes a(y)) = \rho(id_X \otimes a)(x \otimes y)$

$\Rightarrow \omega = \rho(id_X \otimes a)$.





$\Rightarrow$ ω has inverse Ω.

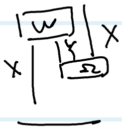
"(3) $\Rightarrow$ (1)" ✓

"(1) $\Rightarrow$ (2)(3)" ω has inverse $\Omega: \mathbb{k} \to Y \otimes X$. $\quad$ $\Omega(1_{\mathbb{k}}) = \sum_{i=1}^{n} e_i' \otimes e_i$.

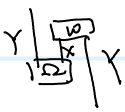
$e_i' \in Y$, $e_i \in X$.

$\underline{b}: X \to Y^*$, $\underline{a}: Y \to X^*$.

Set $l_i := a(\underline{e}'_i) \in X^*$, $l'_i = b(e_i) \in Y^*$. $1 \le i \le n$.

 $= \mid X$, $\Rightarrow \forall x \in X$, $\underline{\sum_{i=1}^{n} l_i(x) e_i = x}$.

$\Rightarrow X, Y$ are projective of finite type

 $= \mid Y$ $\Rightarrow \forall y \in Y$, $\sum_{i=1}^{n} l'_i(y) e'_i = y$.

Set $a' : X^* \to Y$, $b' : Y^* \to X$

$f \mapsto \sum_{i=1}^{n} f(e_i) e'_i \in Y$. $\qquad f \mapsto \sum_{i=1}^{n} f(e'_i) e_i \in X$.

$\Rightarrow a'a = \mathrm{id}_Y$, $aa' = \mathrm{id}_{X^*}$, $b'b = \mathrm{id}_X$, $bb' = \mathrm{id}_{Y^*}$ $\Rightarrow a, b$ are isomorphisms. $\square$

Lem: X, Y: free $\mathbb{k}$-mod, $\underline{\omega}: X \otimes_{\mathbb{k}} Y \to \mathbb{k}$ is non-degenerate

$\Longleftrightarrow \exists n \in \mathbb{Z}_{\ge 1}$, s.t. $\operatorname{rank} X = \operatorname{rank} Y = n$ and.

$\exists$ bases $\{x_i\}_{i=1}^{n}$ of X, $\{y_j\}_{j=1}^{n}$ of Y s.t. matrix $\underline{(\omega(x_i \otimes y_j))_{i,j=1}^{n}}$ is invertible.

If ω is $\underline{\text{non-degenerate}}$, then ω has inverse

$$\underline{\Omega}: \mathbb{k} \to Y \otimes_{\mathbb{k}} X$$

$$\underline{1_{\mathbb{k}}} \mapsto \sum_{i,j=1}^{n} \Omega_{ij}\, y_i \otimes x_i, \quad \Omega_{ij} \in \mathbb{k}.$$

where $(\Omega_{ij})_{i,j=1}^{n}$ is inverse of $(\omega(x_i \otimes y_j))_{i,j=1}^{n}$.

Pf: "$\Longrightarrow$" ω has inverse $\Omega: \mathbb{k} \to Y \otimes_{\mathbb{k}} X$.

Then X, Y are projective of finite type. $\operatorname{rank} X, \operatorname{rank} Y < \infty$.

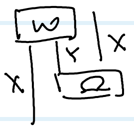
$X \simeq \underline{Y}^* \Rightarrow \operatorname{rank}(X) = \operatorname{rank}(Y^*) = \operatorname{rank}(Y) \overset{\triangle}{=} n$.

Let $\{x_i\}_{i=1}^n$ be base of X, $\{y_i\}_{i=1}^n$ be base of Y

$\{x_i^*\}_{i=1}^n$ be dual base of X^*, $\{y_i^*\}_{i=1}^n$ be dual base of Y^*.

$\in Y\otimes X$

Set $\Omega_{ij} = (y_i^* \otimes x_j^*)\,\Omega(1_{\mathbb{K}}) \in \mathbb{K}$, $1 \le i, j \le n$. $\quad \Omega(1_{\mathbb{K}}) = \sum_{1\le i,j\le n} \boxed{\lambda_{ij}}\, y_i \otimes x_j$

 $= \Big| X$, $\quad (w \otimes id_X)(id_X \otimes_{\mathbb{K}} \Omega) = id_X$

$$x_i \longmapsto x_i \otimes 1_{\mathbb{K}} \mapsto x_i \otimes \Omega(1_{\mathbb{K}}) \mapsto x_i \otimes \Big(\sum_{1\le k,j\le n} \Omega_{kj}\, y_k \otimes x_j\Big)$$

$$\mapsto \sum_{1\le k,j\le n} \Omega_{kj}\, w(x_i \otimes y_k)\, x_j = x_i.$$

$$\Rightarrow \sum_{1\le k\le n} w(x_i \otimes y_k)\,\Omega_{kj} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \ne j \end{cases} \Rightarrow (\Omega_{ij})_{i,j=1}^n \text{ is inverse of } (w(x_i\otimes y_j))_{i,j=1}^n.$$

$(w(x_i \otimes y_j))$.

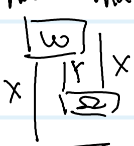
"$\Longleftarrow$" $\operatorname{rank} X = \operatorname{rank} Y = n$, $\{x_i\}_{i=1}^n$ is base of X, $\{y_i\}_{i=1}^n$ is base of Y.

$(\Omega_{ij})_{i,j=1}^n$ is inverse of $(w(x_i \otimes y_j))_{i,j=1}^n$

Def $\Omega : \mathbb{K} \longrightarrow Y \otimes_{\mathbb{K}} X$

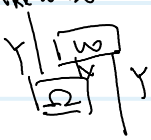
$$1_{\mathbb{K}} \mapsto \sum_{i,j=1}^n \Omega_{ij}\, y_i \otimes_{\mathbb{K}} x_j.$$

Show that

 $= \Big| X$.

$$x_i \longmapsto \sum_{k,j=1}^n \Omega_{kj}\, w(x_i \otimes y_k)\, x_j = x_i$$

Likewise

 $= \Big| Y$. $\qquad \Omega$ is inverse of w.

A category $\mathcal{C}$ is $\mathbb{k}$-linear (or $\mathbb{k}$-category), if $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a left $\mathbb{k}$-mod.
$\forall X, Y \in ob(\mathcal{C})$

s.t the composition of morphisms in $\mathcal{C}$ is $\mathbb{k}$-bilinear

$$(k_1 f_1 + k_2 f_2)\circ g = k_1 (f_1 g) + k_2 (f_2 g)$$

Two $\mathbb{k}$-category $\mathcal{C}, \mathcal{D}$, a functor $F: \mathcal{C} \to \mathcal{D}$ is $\mathbb{k}$-linear

if $\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is $\mathbb{k}$-linear, $\forall X, Y \in ob(\mathcal{C})$.
$f \longmapsto F(f)$

Two $\mathbb{k}$-categories are isomorphic (equivalent), if there is a $\mathbb{k}$-linear isomorphic (equivalent)

A finite set Λ, $\{X_\alpha\}_{\alpha\in\Lambda}$, $X_\alpha \in ob(\mathcal{C})$

def: $D \in ob(\mathcal{C})$ is direct sum of $\{X_\alpha\}_{\alpha\in\Lambda}$, if $\exists$

$$\{p_\alpha : D \to X_\alpha,\ q_\alpha : X_\alpha \to D\}_{\alpha\in\Lambda}$$ coproduct

s.t $\mathrm{id}_D = \sum_{\alpha\in\Lambda} q_\alpha p_\alpha$, $p_\beta q_\alpha = \delta_{\alpha,\beta}\, \mathrm{id}_{X_\alpha}$, $\forall \alpha, \beta \in \Lambda$. $\square$

Prop: If direct sum of $\{X_\alpha\}_{\alpha\in\Lambda}$ exists, then it's unique (up to isom).,
denote it by $\bigoplus_{\alpha\in\Lambda} X_\alpha$.

pf: D', $\{p'_\alpha : D' \to X_\alpha,\ q'_\alpha : X_\alpha \to D'\}_{\alpha\in\Lambda}$ be another direct sum of $\{X_\alpha\}_{\alpha\in\Lambda}$.

Set $f = \sum_{\alpha\in\Lambda} q_\alpha p'_\alpha : D' \to D$ $\Rightarrow$ $fg = \mathrm{id}_D$
$g = \sum_{\alpha\in\Lambda} q'_\alpha p_\alpha : D \to D'$ $\quad gf = \mathrm{id}_{D'}$

$\square$

The sum direct of $\{X_\alpha \mid \alpha\in\Lambda,\ X_\alpha\in ob(Mod_{\Bbbk})\}$ is $\bigoplus_{\alpha\in\Lambda} X_\alpha$.

$p_\beta : \bigoplus_{\alpha\in\Lambda} X_\alpha \to X_\beta$, $q_\beta : X_\beta \to \bigoplus_{\alpha\in\Lambda} X_\alpha$.

$(x_\alpha)_{\alpha\in\Lambda} \mapsto x_\beta$ $\qquad x_\beta \mapsto (y_\alpha)_{\alpha\in\Lambda}$, where $y_\alpha = \begin{cases} x_\beta, & \alpha=\beta \\ 0, & \alpha\neq\beta \end{cases}$.

Prop: $\mathcal{C}$: $\Bbbk$-category. Λ : finite set, $X_\alpha \in ob(\mathcal{C})$ $(\alpha\in\Lambda)$, $Y\in ob(\mathcal{C})$.

Then (1) $Hom_{\mathcal{C}}(Y, \bigoplus_{\alpha\in\Lambda} X_\alpha) \cong \bigoplus_{\alpha\in\Lambda} Hom_{\mathcal{C}}(Y, X_\alpha)$

pf:
$$f \mapsto (p_\alpha f)_{\alpha\in\Lambda}$$
$$\sum_{\alpha\in\Lambda} q_\alpha f_\alpha \leftarrow (f_\alpha)_{\alpha\in\Lambda}$$

(2) $Hom_{\mathcal{C}}(\bigoplus_{\alpha\in\Lambda} X_\alpha, Y) \cong \bigoplus_{\alpha\in\Lambda} Hom_{\mathcal{C}}(X_\alpha, Y)$.

Def: $\mathcal{C}$: $\Bbbk$-category, $X\in ob(\mathcal{C})$ is zero object of $\mathcal{C}$.
if $End_{\mathcal{C}}(X)=0$, i.e. $End_{\mathcal{C}}(X) = id_X$. Denote it by 0.

Prop: If zero object of $\Bbbk$-category $\mathcal{C}$ exists, then it's unique (up to isomor).

Pf. If X, Y are zero objects of $\mathcal{C}$, then

$0_{X,Y} \in End_{\mathcal{C}}(X,Y)$, $0_{Y,X} \in End_{\mathcal{C}}(X,Y)$ zero element.

$0_{X,Y}\, 0_{Y,X} \in End_{\mathcal{C}}(Y,Y) = id_Y$.
$\Rightarrow X \simeq Y$.
$0_{Y,X}\, 0_{X,Y} \in End_{\mathcal{C}}(X,X) = id_X$.

□

Def: $\Bbbk$-category is additive, if it has zero object.
and any finite family of objects of $\mathcal{C}$ has a direct sum in $\mathcal{C}$.

Lem: $\mathcal{C}$: $\mathbb{k}$-cat, $X \in ob(\mathcal{C})$, $End_{\mathcal{C}}(X)$ is a $\mathbb{k}$-algebra with unit element id_X, then following conditions on X are equivalent.

(i) $\mathbb{k} \to End_{\mathcal{C}}(X)$, $k \mapsto k\,id_X$ is a $\mathbb{k}$-mod isomor. $End_{\mathcal{C}}(X) \simeq \mathbb{k}$

(ii) $\mathbb{k} \to End_{\mathcal{C}}(X)$, $k \mapsto k\,id_X$ is a $\mathbb{k}$-algebra isomor.

(iii) the $\mathbb{k}$-algebra $End_{\mathcal{C}}(X)$ is isomorphic to $\mathbb{k}$

(iv) the $\mathbb{k}$-mod $End_{\mathcal{C}}(X)$ is free of rank 1.

Pf: (i) $\Rightarrow$ (iv) ✓ (i) $\Leftrightarrow$ (ii) ✓.

(ii) $\Leftrightarrow$ (iii)

(iv) $\Rightarrow$ (i) Let $f \in End_{\mathcal{C}}(X)$ be $\mathbb{k}$-base of $End_{\mathcal{C}}(X)$, then $\exists k \in \mathbb{k}$ s.t $kf = id_X$, $k \neq 0$, othwise $End_{\mathcal{C}}(X) = 0$, $(\forall g \in End_{\mathcal{C}}(X),\ g = id_X g = (kf)g = 0)$

Thus, $f = k^{-1} id_X$. id_X is base of $End_{\mathcal{C}}(X)$. $\square$

Def, Objects of $\mathcal{C}$ satisfying these condition of the above lemma are said to be simple.

X is simple in $Mod_{\mathbb{k}}$. $End_{\mathbb{k}}(X) \simeq \mathbb{k}$.

Prop: All objects of $\mathcal{C}$ isomorphic a simple object are simple.

Pf: $X \in ob(\mathcal{C})$ simple. $X \simeq Y$, $\exists f: X \to Y$, $g: Y \to X$. s.t $gf = id_X$, $fg = id_Y$

Set $\varphi : \overset{\cong \mathbb{K}}{\mathrm{End}_{\mathcal{C}}(X)} \longrightarrow \mathrm{End}_{\mathcal{C}}(Y)$

$h \longmapsto fhg \in \mathrm{End}_{\mathcal{C}}(Y)$ <u>$\mathbb{K}$-linear.</u>

its inverse $\psi : \mathrm{End}_{\mathcal{C}}(Y) \longrightarrow \mathrm{End}_{\mathcal{C}}(X)$

$p \mapsto gpf$.

$\psi\varphi = \mathrm{id}$, $\varphi\psi = \mathrm{id}$. $\square$