$F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{E}$ monoidal functor.

Def $GF: \mathcal{C} \to \mathcal{E}$ with

$$(GF)_0 = G(F_0)G_0, \quad (GF)_2(X,Y) = G(F_2(X,Y))\,G_2(F(X),F(Y)).$$

$$(GF)_0 : 1 \xrightarrow{G_0} G(1) \xrightarrow{G(F_0)} GF(1)$$

$$(GF)_2(X,Y) : GF(X) \otimes GF(Y) \xrightarrow{G_2(F(X),F(Y))} G(F(X) \otimes F(Y)) \xrightarrow{G(F_2(X,Y))} GF(X \otimes Y).$$

$F: \mathcal{C} \to \mathcal{D}$ strong monoidal functor

Def: $F^{rev}: \mathcal{C}^{rev} \to \mathcal{D}^{rev}$, with

$$(F^{rev})_0 = F_0^{-1}, \quad (F^{rev})_2(X,Y) = F_2(Y,X)^{-1}.$$

$\mathcal{C}$ pivotal category, double dual functor $?^{**}: \mathcal{C} \to \mathcal{C}$

with $X \to X^{**} = (X^*)^*$, $f \mapsto f^{**} = (f^*)^*$.

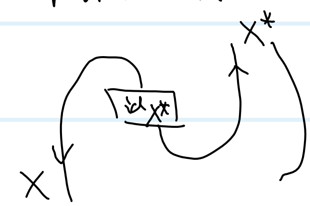
Recall: $?^*: \mathcal{C}^{rev} \to \mathcal{C}$ strong monoidal functor

$(?^*)^{rev}: (\mathcal{C}^{rev})^{rev} = \mathcal{C} \to \mathcal{C}^{rev}$,

Then $?^{**} = ?^* \circ (?^*)^{rev}: \mathcal{C} \to \mathcal{C}$ is strong monoidal functor.

Thm: The pivotal structure $\psi: 1_{\mathcal{C}} \to ?^{**}$ is monoidal.

$\psi_X : X \to X^{**}$



natural isomorphism.

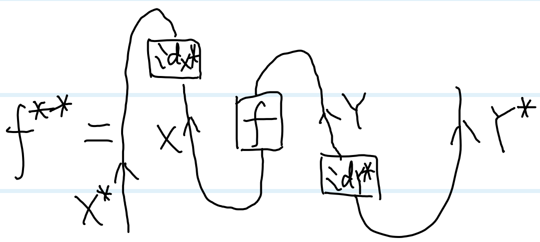
pf: $\underline{\psi \text{ is monoidal}} \iff$ (1) $\psi_1 = (?^{**})_0$ and

(2) $\psi_{X\otimes Y} = (?^{**})_2(X,Y)(\psi_X \otimes \psi_Y)$, $\forall X, Y \in Ob(\mathcal{C})$

natural

$$\boxed{\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\psi_X"'] & Y \arrow[d, "\psi_Y"] \\ X^{**} \arrow[r, "f^{**}"] & Y^{**} \end{tikzcd}}$$

$f^{**} =$



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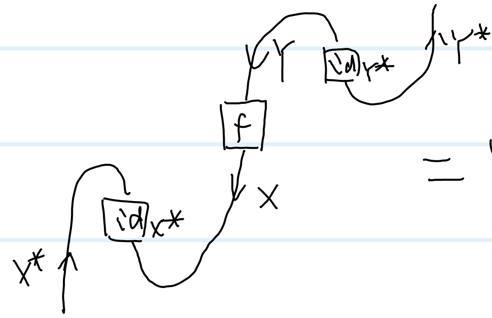


$= \psi_Y f \psi_X^{-1}$

Note that

$$(?^{**})_0 = ((?^*)_0^*)^{-1}(?^*)_0$$

$$(?^{**})_2(X,Y) = ((?^*)_2(Y,X)^*)^{-1}(?^*)_2(X^*,Y^*)$$

Need to show that.

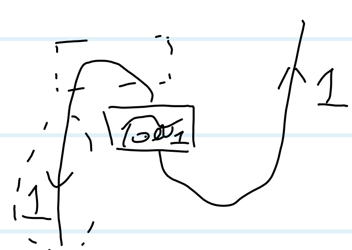
(1) $(?^*)_0^* \psi_1 = (?^*)_0$

(2) $(?^*)_2(Y,X)^* \psi_{X\otimes Y} = (?^*)_2(X^*,Y^*)(\psi_X \otimes \psi_Y)$

$(?^*)_0 = \mathrm{coev}_1 = \widetilde{\mathrm{coev}}_1$

$(?^*)_0^* \psi_1 =$

$=$



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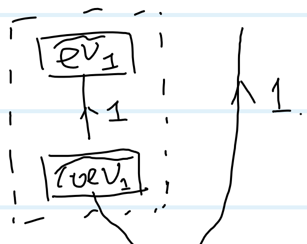

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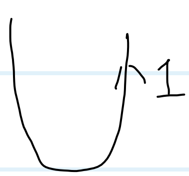
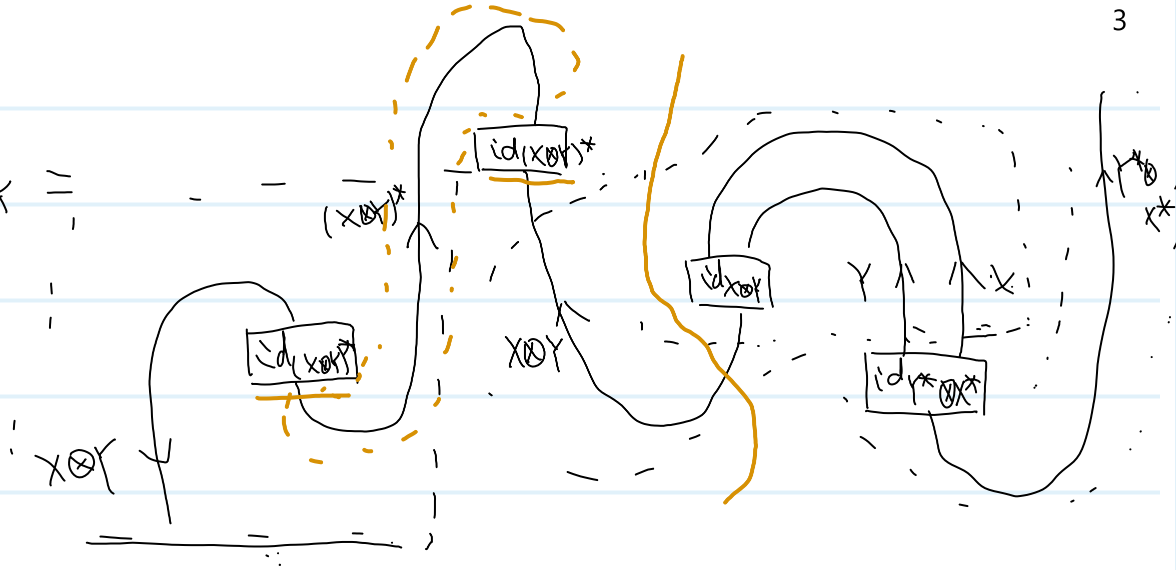


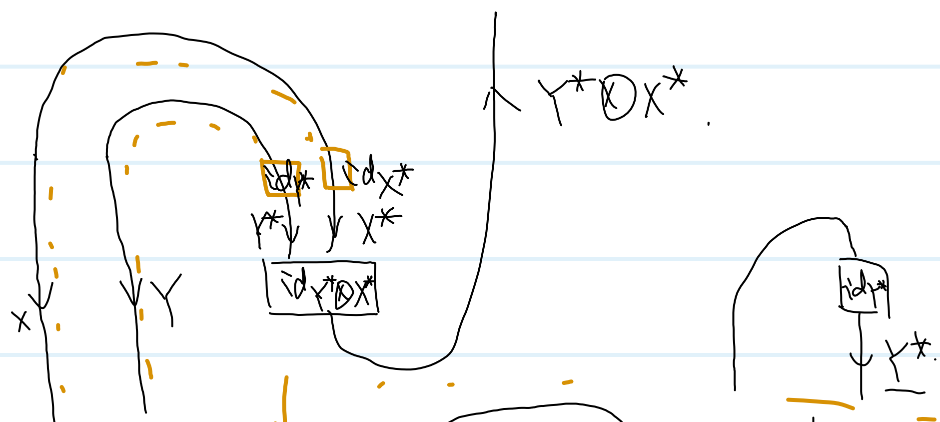
$= \widetilde{\mathrm{coev}}_1 = (?^*)_0$.

$(?^*)_2(Y,X)^* \varphi_{X\otimes Y} =$



$((?^*)_2(Y,X))^* = Y^* \otimes X^* \to (X\otimes Y)^*$

$=$



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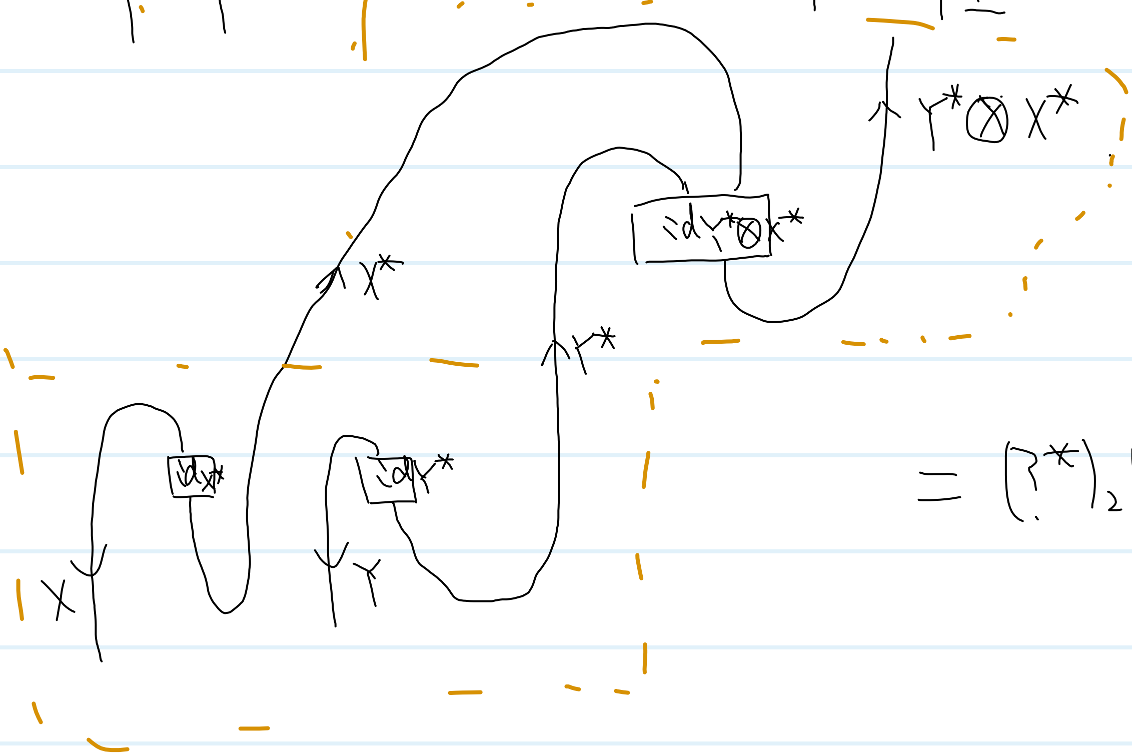


$= (?^*)_2(X^*,Y^*)(\varphi_X \otimes \varphi_Y)$

$\mathcal{C}$ pivotal category.

For $X \in \mathrm{ob}(\mathcal{C})$, $\varepsilon \in \{+,-\}$, def $X^{\varepsilon} = \begin{cases} X & \varepsilon = + \\ X^* & \varepsilon = - \end{cases}$.

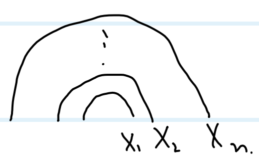
For $\underline{S} = ((X_1, \varepsilon_1), \cdots, (X_n, \varepsilon_n))$, $X_i \in \mathrm{ob}(\mathcal{C})$, $\varepsilon_i \in \{+,-\}$

def $X_S = X_1^{\varepsilon_1} \otimes \cdots \otimes X_n^{\varepsilon_n} \in \mathrm{ob}(\mathcal{C})$

For $S = \phi$, set $X_\phi = \mathbb{1}$

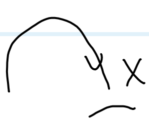
def $S^* = ((X_1, -\varepsilon_1), \cdots, (X_n, -\varepsilon_n))$, $\phi^* = \phi$.

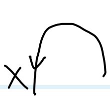
Set

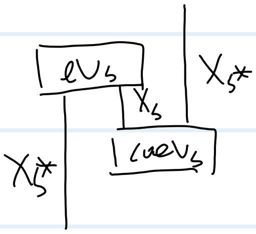
$\underline{ev_S}$ =  $X_{S^*} \otimes X_S \to \mathbb{1}$.

$\underline{coev_S}$ =  $\mathbb{1} \to X_S \otimes X_{S^*}$.

Here the arc labeled with X_i is oriented toward right endpoint if $\varepsilon_i = +$ and toward the left endpoint if $\varepsilon_i = -$.

$ev_{(X,+)}$ =  $= ev_X$ $\qquad$ $coev_{(X,+)} = coev_X$.

$ev_{(X,-)}$ =  $= \widetilde{ev}_X$ $\qquad$ $coev_{(X,-)} = \widetilde{coev}_X$.



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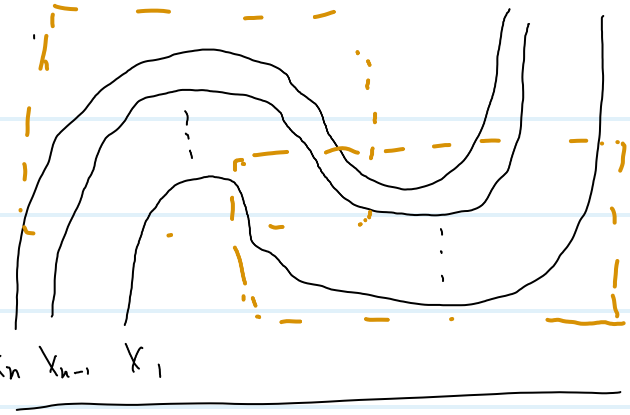
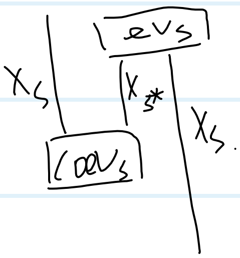


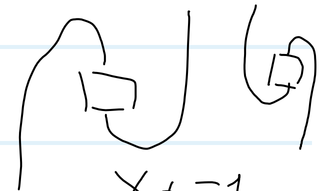
$= id_{X_{s^*}}$.



$= id_{X_s}$.

ev_s is non-degenerate with inverse $coev_s$.

ψ_X



$X_s \longrightarrow (X_{\underline{s^*}})^*$.

$X_\phi = \underline{1}$

$s = \phi$ $\quad \phi^* = \phi$.

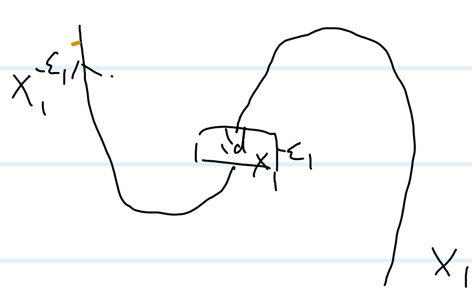
Set $\psi_\phi = coev_{\underline{1}} = \widetilde{coev}_{\underline{1}} : \underline{1} \to \underline{1}^*$.

For $s = ((X_1, \varepsilon_1), \cdots (X_n, \varepsilon_n))$ Def

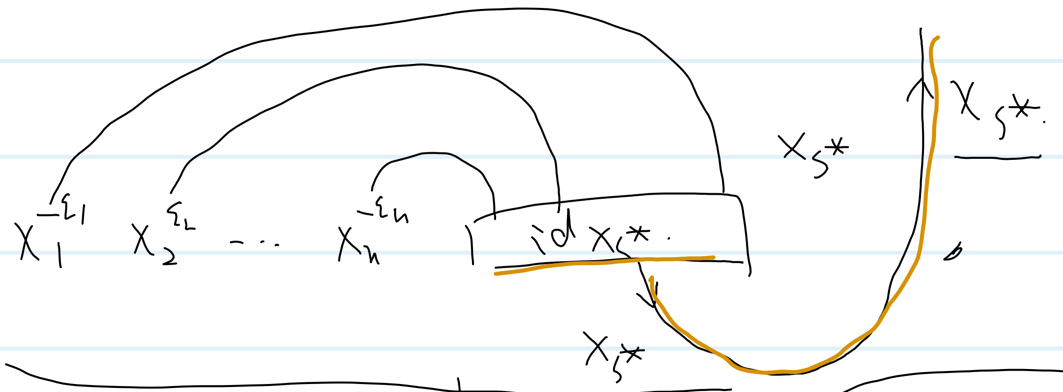
$\psi_s : X_s \to (X_{s^*})^*$. as composition of two map

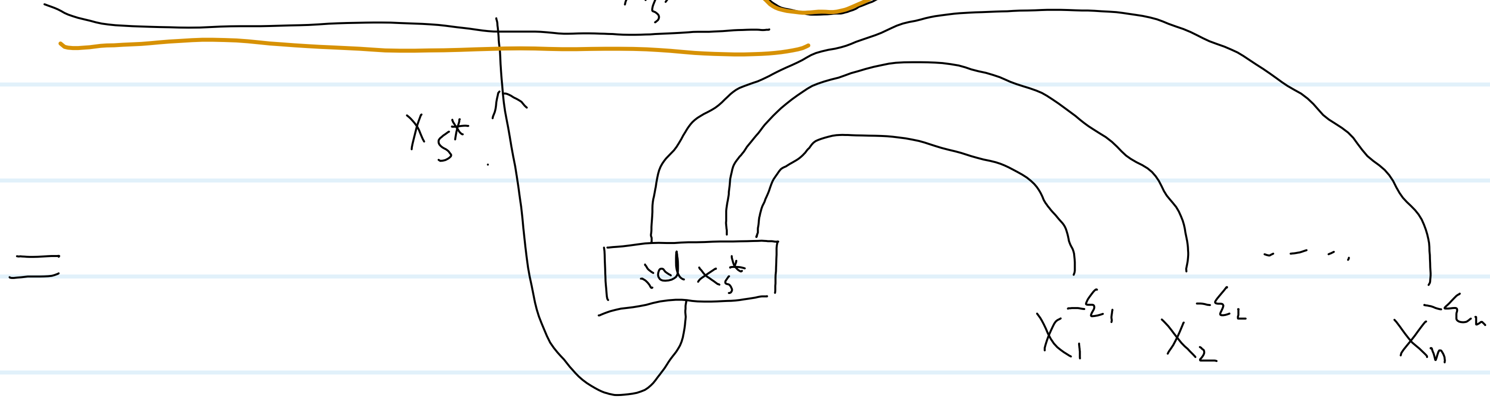
1. $X_s = X_1^{\varepsilon_1} \otimes \cdots \otimes X_n^{\varepsilon_n} \longrightarrow (\underline{X_1^{-\varepsilon_1}})^* \otimes \cdots \otimes (\underline{X_n^{-\varepsilon_n}})^*$.

$n=1,\ \varepsilon_1 = +$. $\qquad X_1 \to X_1^{**}$.



2. $(X_1^{-\varepsilon_1})^* \otimes \cdots \otimes (X_n^{-\varepsilon_n})^* \longrightarrow (X_n^{-\varepsilon_n} \otimes \cdots) \otimes X_1^{-\varepsilon_1})^* = (X_{\varsigma^*})^*$

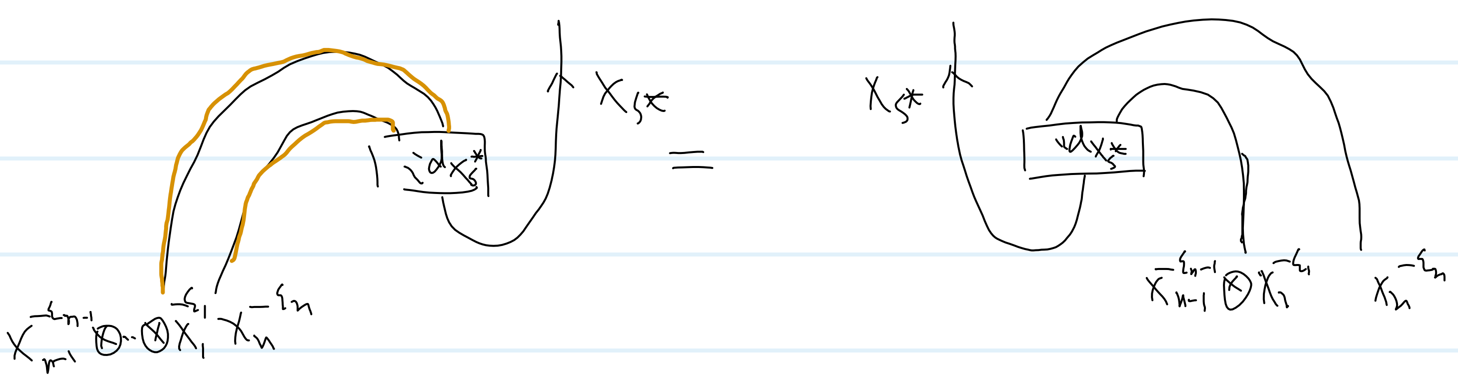




Use induction on n

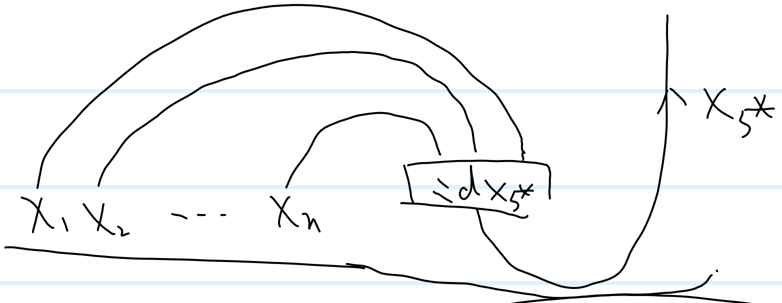
$n=1,2$ ✓

$(X_{n-1}^{-\varepsilon_{n-1}} \otimes \cdots \otimes X_1^{-\varepsilon_1})^* \otimes (X_n^{-\varepsilon_n})^* \longrightarrow (X_{\varsigma^*})^*$.



i]

$\Phi_s =$

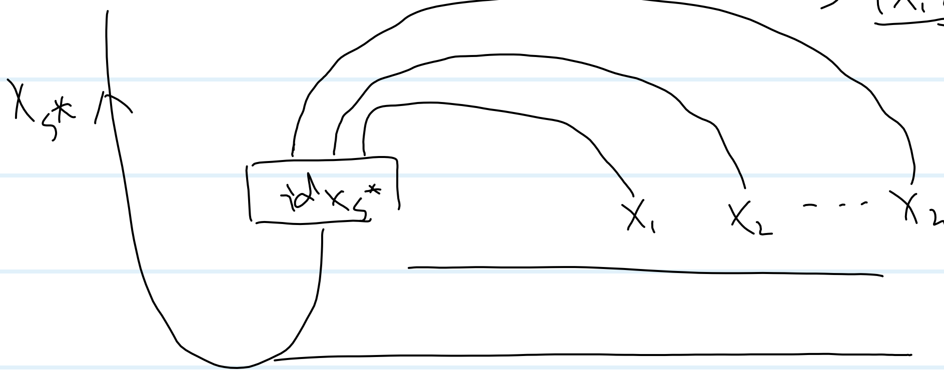


$X_s \longrightarrow ((X_s*))^*$

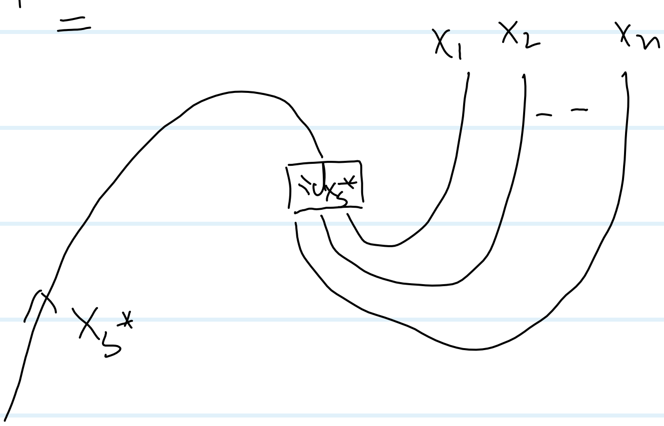
$S\ (X_1\ \varepsilon_1) \cdots (X_n\ \varepsilon_n)^*$

$(X_1, -\varepsilon_1) \cdots (X_n, -\varepsilon_n)$

$=$



$\Phi_s^{-1} =$



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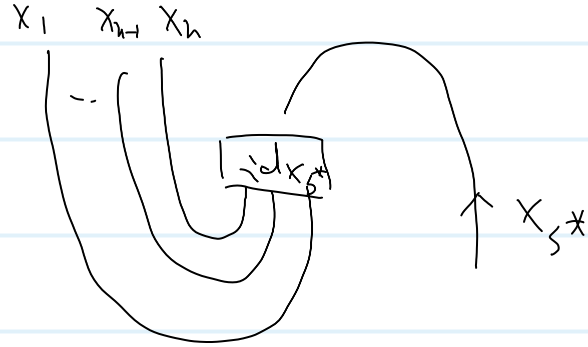
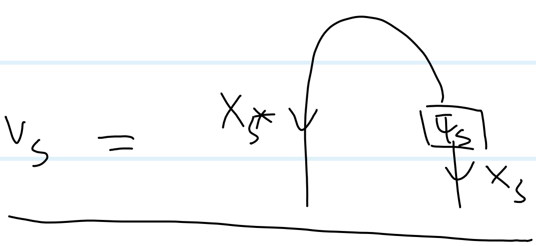
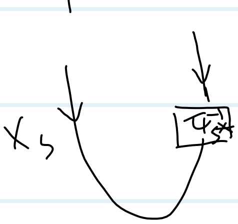


Lem: $\underline{\Phi_s}: X_s \to (X_s*)^*, \quad \underline{\Phi_{s*}}: X_s* \longrightarrow (X_s)^*.$

$=$

$= ev_s =$





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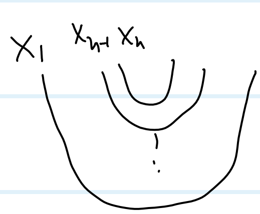
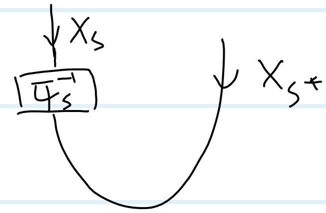


$= coev_s =$

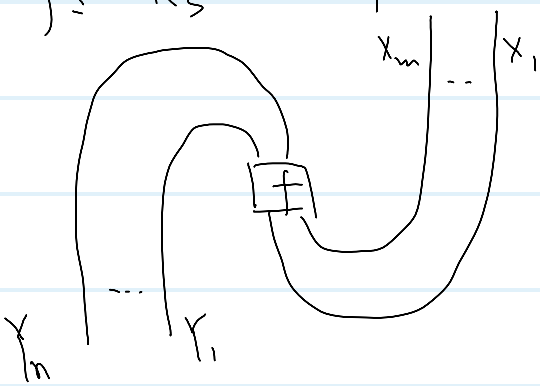


Lem: For any $S = ((X_1, \varepsilon_1) \cdots (X_m, \varepsilon_m))$

$T = ((Y_1, \nu_1) \cdots (Y_n, \nu_n))$

$f : X_S \to X_T$

Then



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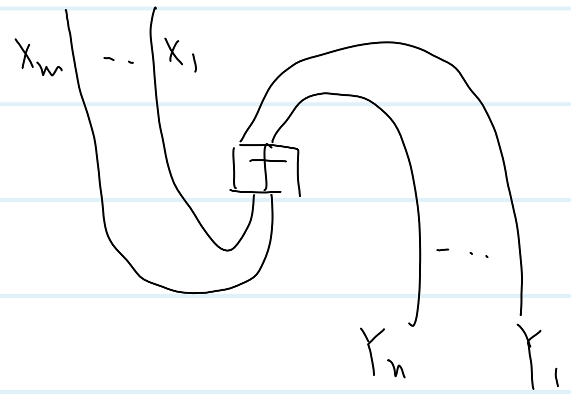
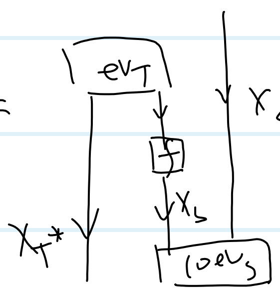


$X_{T^*} \to X_{S^*}$. $\qquad$ $X_{T^*} \to X_{S^*}$.

Pf: The left =



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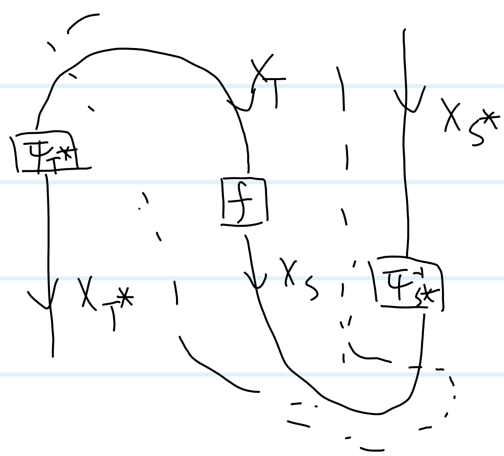

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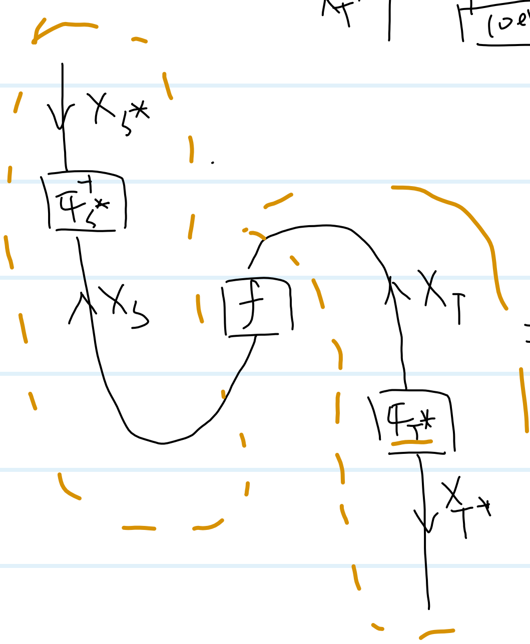

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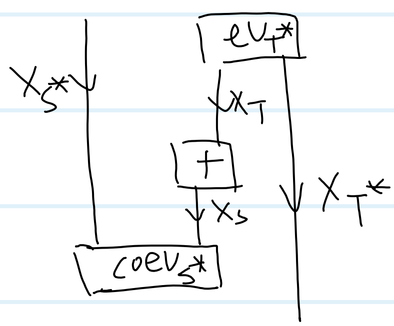
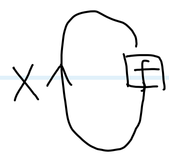
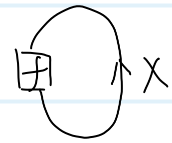


= The right.

$\mathcal{C}$ pivotal category, $X \in ob(\mathcal{C})$, $f \in End_{\mathcal{C}}(X)$.

Def left trace of f $\quad tr_l(f) =$ 

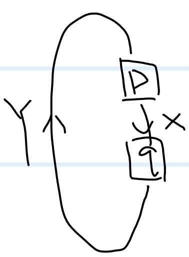
$\in End_{\mathcal{C}}(1)$

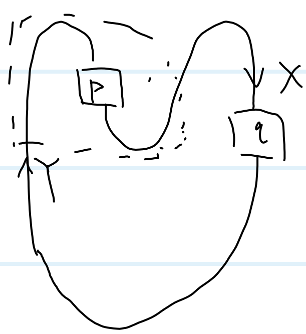
right trace of f $\quad tr_r(f) =$ 

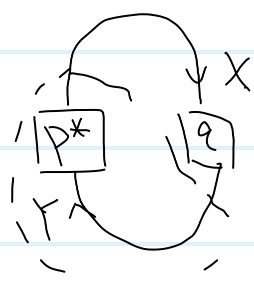
$\in End_{\mathcal{C}}(1)$.

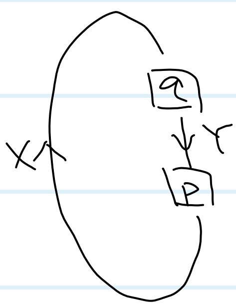
Prop: $\forall\ p : X \to Y$, $q : Y \to X$, then

$$tr_l(pq) = tr_l(qp), \quad tr_r(pq) = tr_r(qp).$$

pf: $tr_l(pq) =$ 

$=$ 

$=$ 

$\stackrel{dual}{=}$ 

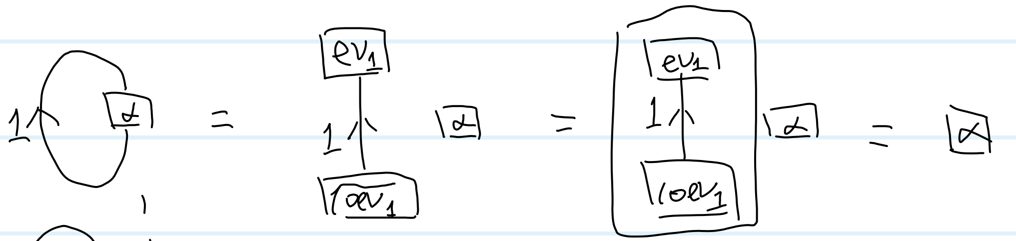
$= tr_l(qp)$.

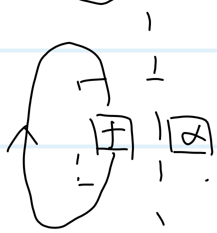
$\forall\ \alpha \in End_{\mathcal{C}}(1)$, $f \in End_{\mathcal{C}}(X)$, $g \in End_{\mathcal{C}}(Y)$. Then

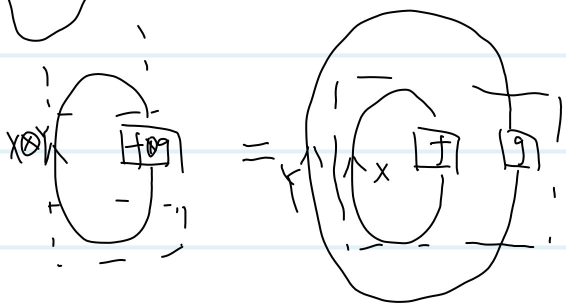
$tr_l(\alpha) = tr_r(\alpha) = \alpha$, $\quad tr_l(f \cdot \alpha) = \alpha\, tr_l(f)$, $\quad tr_r(\alpha \cdot f) = \alpha\, tr_r(f)$.

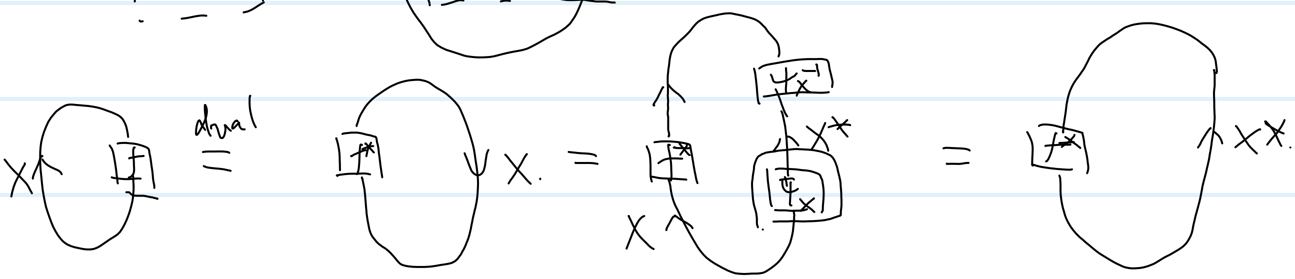
$tr_l(f \otimes g) = \underline{tr_l(tr_l(f) \cdot g)}$, $\quad tr_l(f) = tr_r(f^*)$.

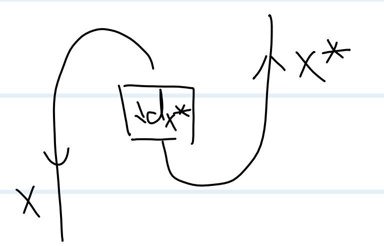
$tr_r(f \otimes g) = tr_r(f \cdot tr_r(g))$, $\quad tr_r(f) = tr_l(f^*)$.

Pf: $tr_l(\alpha) =$  $= \alpha$.

$tr_l(f\cdot\alpha) =$  $= tr_l(f)\cdot\alpha = \alpha\, tr_l(f)$.

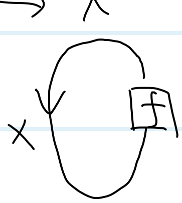
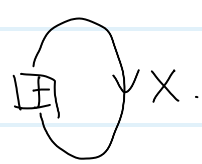
$tr_l(f\otimes g) =$  $= tr_l(tr_l(f)\cdot g)$

$tr_l(f) =$ 

$\psi_X :$  ψ_X^{-1}

$tr_l(f) = tr_l(f^{**})$, $tr_r(f) = tr_r(f^{**})$

For $f: X^* \to X^*$.

$tr_l(f) =$  $tr_r(f) =$ 

Def left dimension $\dim_l(X) = tr_l(id_X)$ $X \in ob(\mathcal{C})$

right $\dim_r(X) = tr_r(id_X)$ $X \in ob(\mathcal{C})$.

A pivotal category is <u>spherical</u>, if $tr_r(f) = tr_l(f)$, $\forall f \in End_{\mathcal{C}}(X)$.

Def trace of f: $tr(f) = tr_l(f) = tr_r(f)$. $\square$