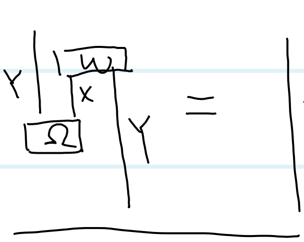
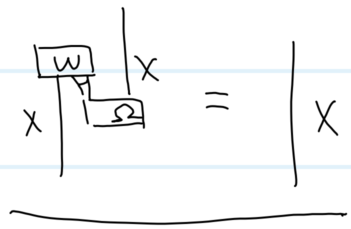
$\mathcal{C}$ monoidal category.

Pairing between $X, Y \in \mathrm{ob}(\mathcal{C})$ is morphism $X \otimes Y \to 1$.

Pairing $\omega : X \otimes Y \to 1$ is non-degenerate, if $\exists\ \Omega : 1 \to Y \otimes X$ s.t.



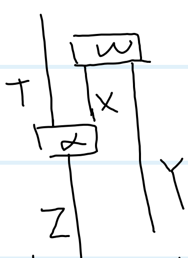
and



Ω is called inverse of ω.

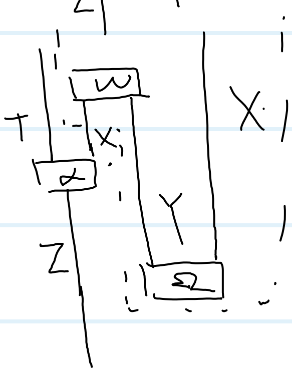
Prop: Pairing $\omega : X \otimes Y \to 1$ its inverse $\Omega : 1 \to Y \otimes X$.

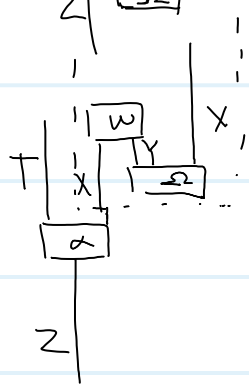
Then $\mathrm{Hom}_{\mathcal{C}}(Z, T \otimes X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Z \otimes Y, T)$ is bijection.

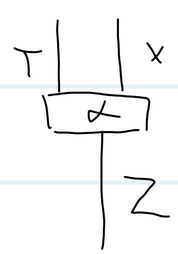
$\alpha \longmapsto$ 

and

$\longleftarrow \beta$.

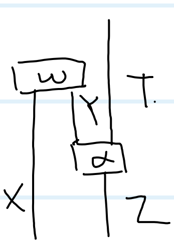
$\alpha \longmapsto$ 

$=$ 

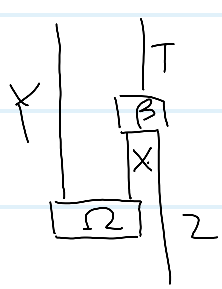
$=$ 

$= \alpha$.

$\mathrm{Hom}_{\mathcal{C}}(Z, Y \otimes T) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X \otimes Z, T)$ is bijection.

$\alpha \longmapsto$ 

and



$\longleftarrow \beta$

Lem: $F : \mathcal{C} \to \mathcal{D}$ strong monoidal functor. (F_0, F_2, F).

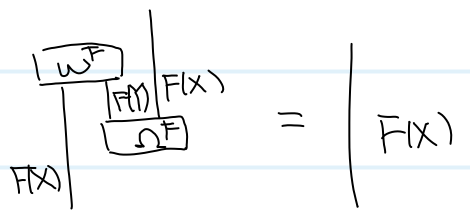
Pairing $\omega : X \otimes Y \to 1$ its inverse $\Omega : 1 \to Y \otimes X$ in $\mathcal{C}$.

Then we have.

Pairing $\omega^F : F(X) \otimes F(Y) \xrightarrow{F_2(X,Y)} F(X \otimes Y) \xrightarrow{F(\omega)} F(1) \xrightarrow{F_0^{-1}} 1$.

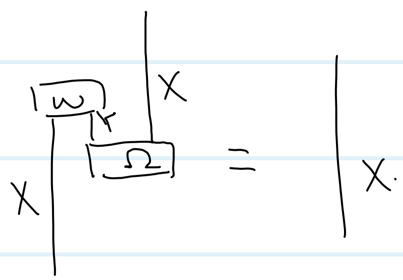
its inverse Ω^F: $1 \xrightarrow{F_0} F(1) \xrightarrow{F(\Omega)} F(Y \otimes X) \xrightarrow{F_2(Y,X)^{-1}} F(Y) \otimes F(X)$

Pf: show that.



and

We know that.



and

$id_X = X \to X \otimes 1 \xrightarrow{id_X \otimes \Omega} X \otimes (Y \otimes X) \to (X \otimes Y) \otimes X \xrightarrow{\omega \otimes id_X} 1 \otimes X \to X.$

F

$\mathrm{id}_{F(X)} =$

$$\begin{tikzcd}
F(X) \arrow[d, "F(r_X)^{-1}"] & F(X)\otimes 1 \arrow[l, "r^{-1}_{F(X)}"'] \arrow[d, "\mathrm{id}_{F(X)}\otimes F_0"] & \\
F(X\otimes 1) \arrow[d, "F(\mathrm{id}_X\otimes\Omega)"] & F(X)\otimes F(1) \arrow[l, "F_2(X,1)"'] \arrow[d, "\mathrm{id}_{F(X)}\otimes F(\Omega)"] & \\
F(X\otimes(Y\otimes X)) \arrow[d, "F(a_{X,Y,X})^{-1}"] & F(X)\otimes F(Y\otimes X) \arrow[l, "F_2(X,Y\otimes X)"'] \arrow[r, "\mathrm{id}_{F(X)}\otimes F_2(Y,X)^{-1}"] & F(X)\otimes(F(Y)\otimes F(X)) \arrow[d, "a^{-1}_{F(X),F(Y),F(X)}"] \\
F((X\otimes Y)\otimes X) \arrow[r] \arrow[d, "F(w\otimes \mathrm{id}_X)"] & F(X\otimes Y)\otimes F(X) \arrow[d, "F(w)\otimes\mathrm{id}_{F(X)}"] & (F(X)\otimes F(Y))\otimes F(X) \arrow[l, "F_2(X,Y)\otimes \mathrm{id}_{F(X)}"'] \\
F(1\otimes X) \arrow[r, "F_2(1,X)^{-1}"] \arrow[d, "F(l_X)"] & F(1)\otimes F(X) \arrow[d, "F_0^{-1}\otimes\mathrm{id}_{F(X)}"] & \\
F(X) \arrow[r, "l_{F(X)}"] & 1\otimes F(X) &
\end{tikzcd}$$

$\square$

Left dual of $X \in \mathrm{ob}(\mathcal{C})$ is ${}^{\vee}X \in \mathrm{ob}(\mathcal{C})$ and

Pairing $ev_X : {}^{\vee}X \otimes X \to 1$ its inverse $coev_X : 1 \longrightarrow X \otimes {}^{\vee}X$.

Right dual of $X \in \mathrm{ob}(\mathcal{C})$ is $X^{\vee} \in \mathrm{ob}(\mathcal{C})$ and

Pairing $\widetilde{ev}_X : X \otimes X^{\vee} \to 1$ its inverse $\widetilde{coev}_X : 1 \longrightarrow X^{\vee} \otimes X$.

Def: Monoidal cate- $\mathcal{C}$ is left (right) rigid, if

$\forall X \in \mathrm{ob}(\mathcal{C})$ has left (right) dual.

is rigid, if $\forall X \in \mathrm{ob}(\mathcal{C})$ has left and right dual.

$\mathcal{C}$ category, Def $\mathcal{C}^{op}$.

$ob(\mathcal{C}^{op}) = ob(\mathcal{C})$, $Hom_{\mathcal{C}^{op}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$

$\mathcal{C}$ monoidal category, Def $\mathcal{C}^{rev}$

$\shortparallel$

$(\mathcal{C}, 1, \otimes, a, l, r)$ $= (\mathcal{C}^{op}, \otimes^{op}, 1, a^{rev}, l^{rev}, r^{rev})$.

where. $X \otimes^{op} Y = Y \otimes X$, $(a^{rev})_{X,Y,Z} = a_{Z,Y,X}$.

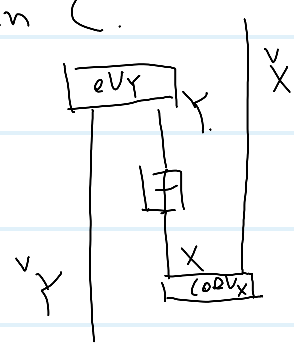
$(l^{rev})_X = r_X^{-1}$, $(r^{rev})_X = l_X^{-1}$.

$\mathcal{C}$ left rigid category, def left dual functor.

$^\vee ? : \mathcal{C}^{rev} \to \mathcal{C}$.

(1) $X \in ob(\mathcal{C}^{rev}) \longmapsto {}^\vee X$.

(2) $f : X \to Y$ in $\mathcal{C}$. ${}^\vee Y \longrightarrow {}^\vee X$.

$^\vee f =$ 

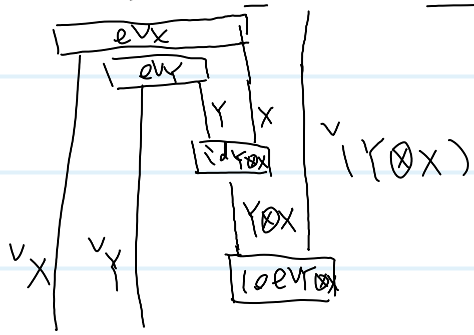
$: {}^\vee Y \longrightarrow {}^\vee X$.

$^\vee ?$ is strong monoidal functor.

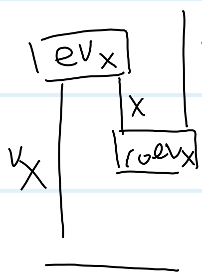
$({}^\vee ?)_0 = coev_1 : 1 \longrightarrow {}^\vee 1$ $1 \xrightarrow{coev_1} 1 \otimes {}^\vee 1 \longrightarrow {}^\vee 1$.

$\forall X, Y \in \mathrm{obj}(\mathcal{C}^{rev})$.

$$^\vee\xi_2(X,Y) : {}^\vee X \otimes {}^\vee Y \to {}^\vee(Y \otimes X)$$



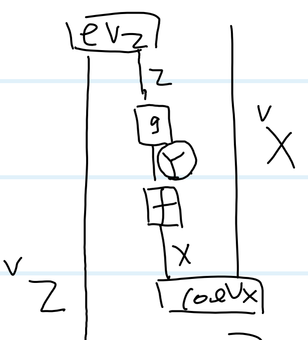
pf: $^\vee id_X =$



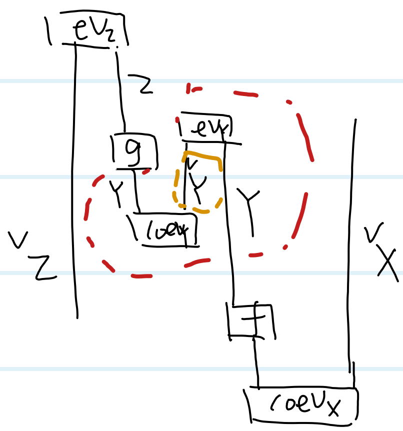
$= \Big| \; {}^\vee X = id_{^\vee X}$.

$X \xrightarrow{f} Y \xrightarrow{g} Z$, $\quad {}^\vee(gf) = ({}^\vee f)({}^\vee g)$.

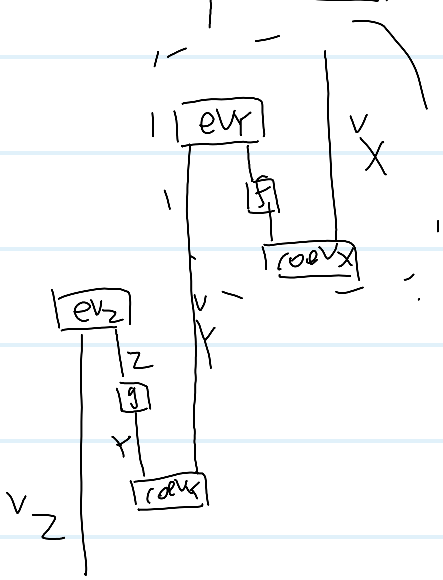
$^\vee(fg) =$



$=$



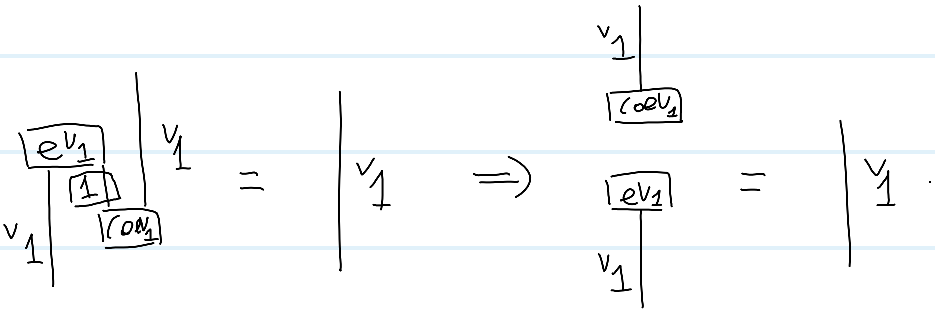
$=$

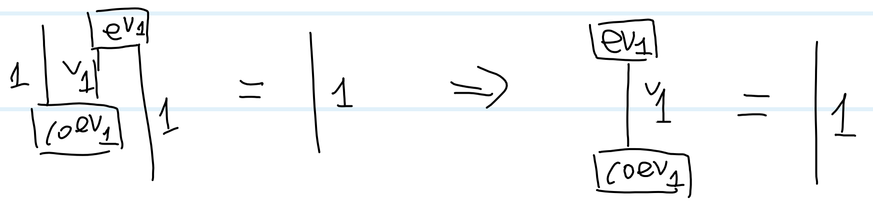


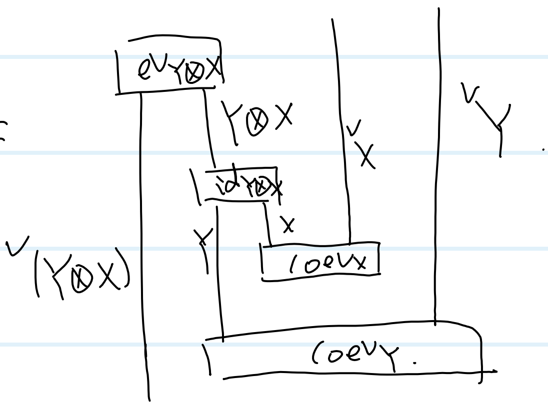
$= ({}^\vee f)({}^\vee g)$.

$coev_1 : 1 \to 1 \otimes {}^\vee 1 \to {}^\vee 1$, $\quad \underline{(coev_1)^\dagger = ev_1} : {}^\vee 1 \to 1$

${}^\vee 1 \to {}^\vee 1 \otimes 1 \xrightarrow{ev_1} 1$



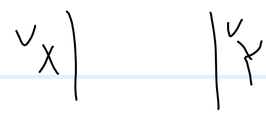


$({}^\vee ?_2(X,Y))^\dagger =$ 

${}^\vee X \otimes {}^\vee Y \xrightarrow{{}^\vee ?_2(X,Y)} {}^\vee(Y \otimes X)$

$\underline{{}^\vee(Y \otimes X) \to {}^\vee X \otimes {}^\vee Y}$.

$\square$

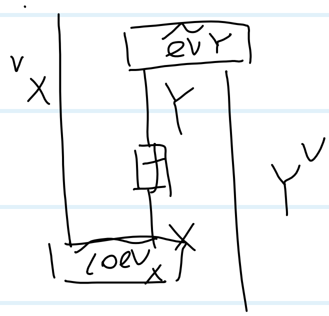


$\mathcal{C}$ right rigid category def right dual functor

$?^\vee : \mathcal{C}^{rev} \to \mathcal{C}$

(1) $X \mapsto X^\vee$.

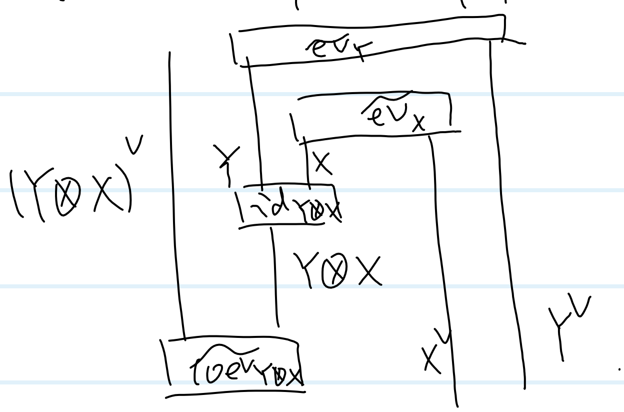
(2) $f : X \to Y$.

$f^\vee =$  $: Y^\vee \to X^\vee$.

$?^\vee$ is strong monoidal functor.

$(?^\vee)_0 = \widetilde{coev}_1 : 1 \to 1^\vee$

$?^\vee_2(X, Y) : X^\vee \otimes Y^\vee \to (Y \otimes X)^\vee$.



□

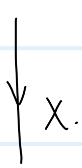
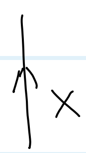
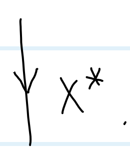
Def: A <u>pivotal category</u> $\mathcal{C}$ is rigid category and $?^\vee = {}^\vee?$ as monoidal functor.

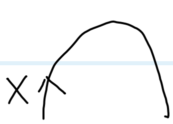
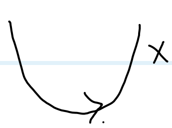
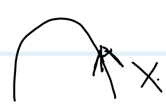
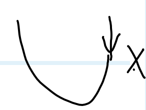
$\forall X \in ob(\mathcal{C})$, $X^* = {}^\vee X = X^\vee$ is called dual of X.

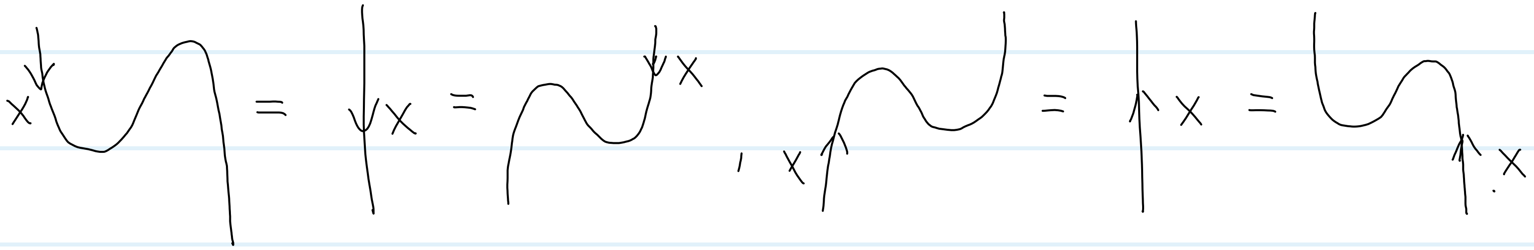
$ev_X : \underline{X^*} \otimes X \to 1$ $\quad coev_X : 1 \to X \otimes X^*$

$\widetilde{ev}_X : X \otimes X^* \to 1$ $\quad \widetilde{coev}_X : 1 \to X^* \otimes X$.

Penrose diagram in pivotal category.

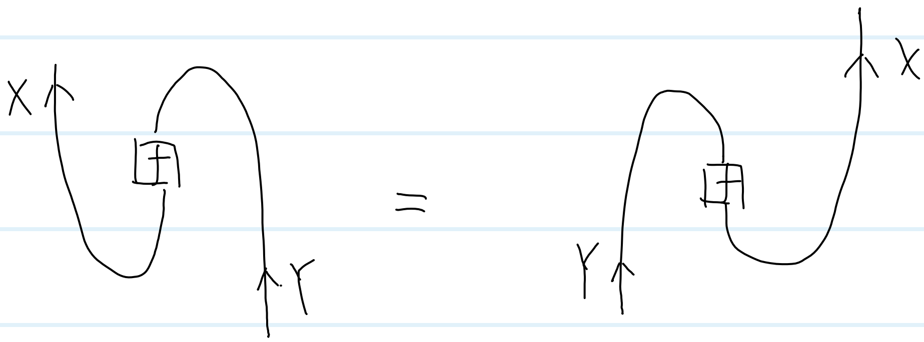
$id_X =$  $\quad id_{X^*} =$  $=$ 

$\underline{ev_X} :$  $\quad \underline{coev_X} :$  $\quad \widetilde{ev}_X :$  $\quad \widetilde{coev}_X :$ 



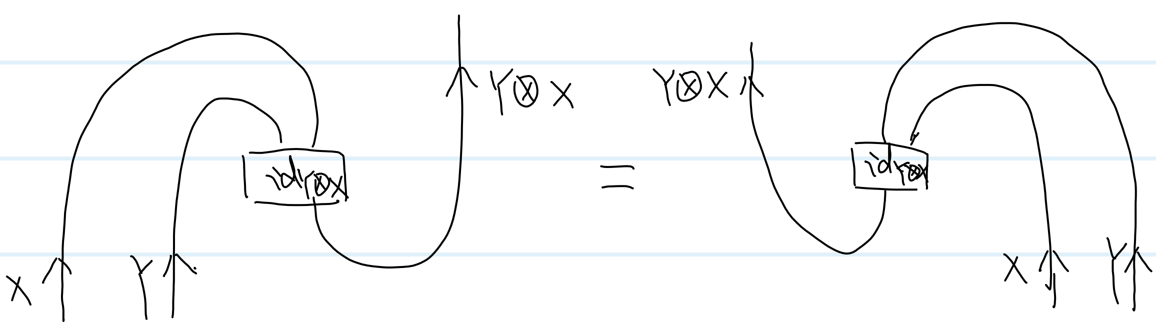
$\forall f : X \to Y$ in $\mathcal{C}$.

$f^* = {}^\vee f = f^\vee : Y^* \to X^*$

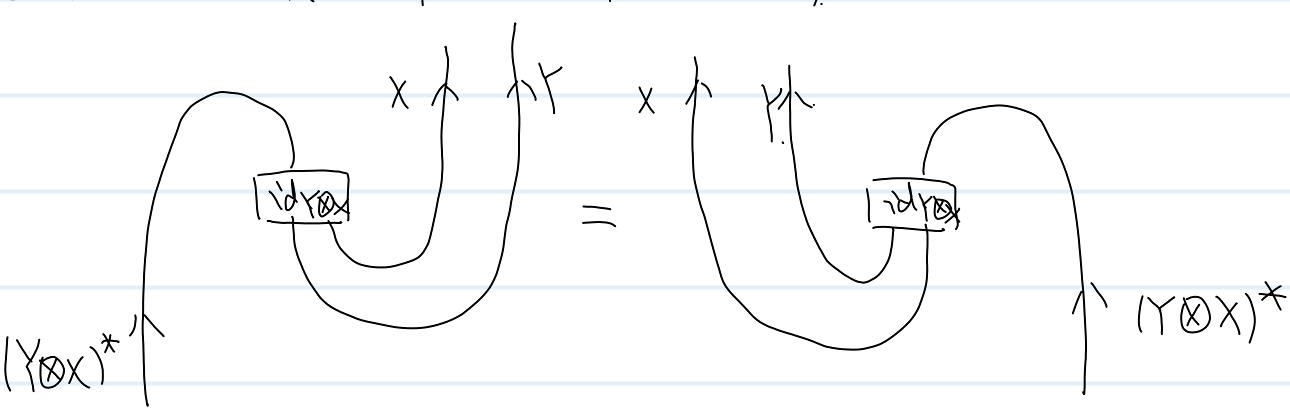


$coev_1 = \widetilde{coev}_1 : 1 \to 1^* \Rightarrow ev_1 = \widetilde{ev}_1 : 1^* \to 1$.

$X^* \otimes Y^* \longrightarrow (Y \otimes X)^*$, $X, Y \in ob(\mathcal{C})$

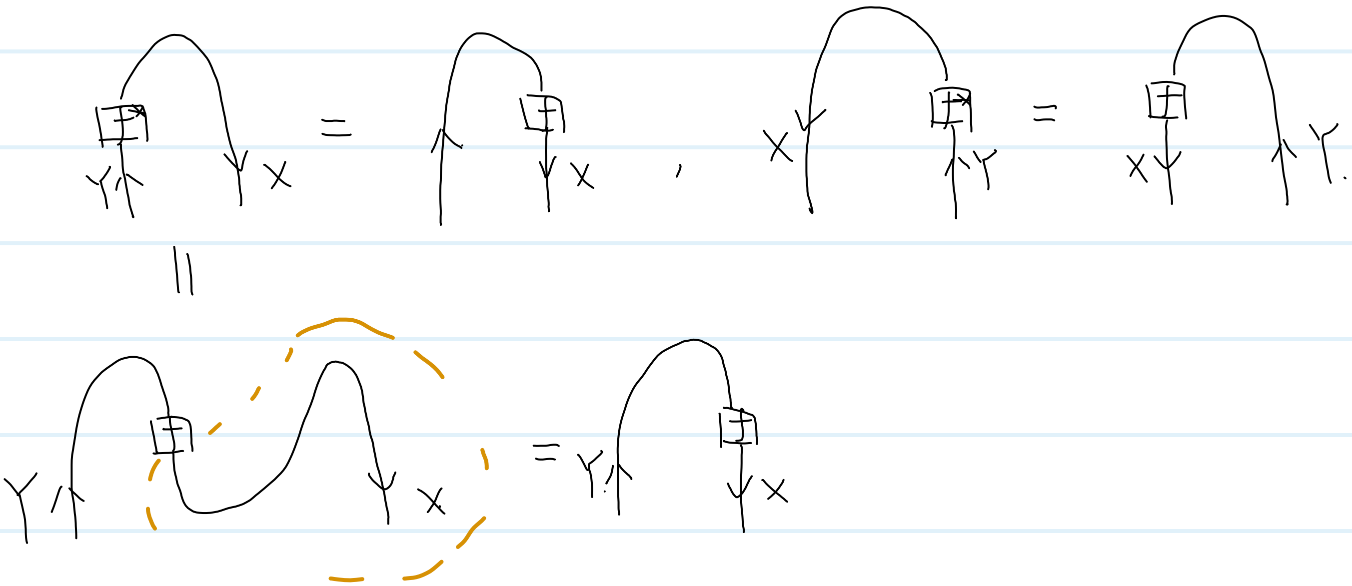


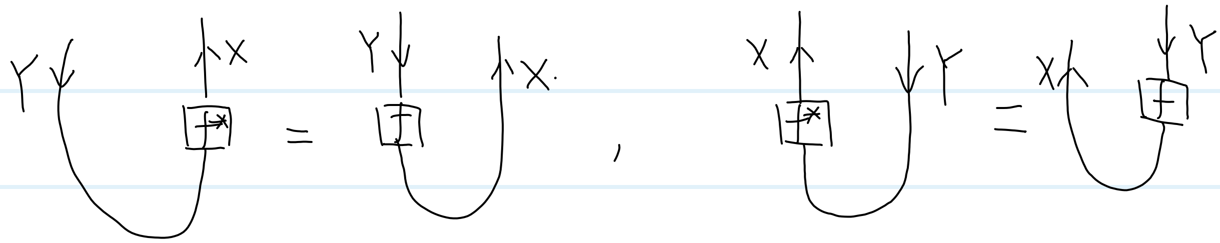
$(Y \otimes X)^* \longrightarrow X^* \otimes Y^*$, $X, Y \in ob(\mathcal{C})$.



dual morphism identities

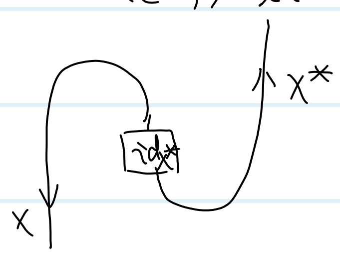
$f: X \to Y$, $f^*: Y^* \to X^*$.



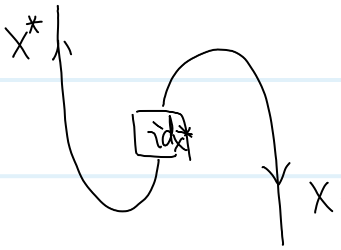
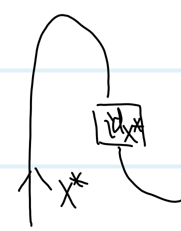
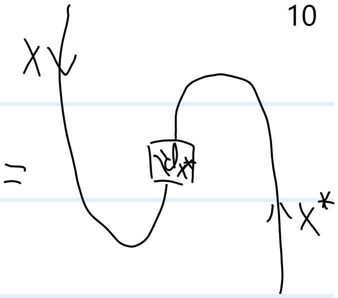


$\mathcal{C}$ pivotal category.

Double dual. $\forall X \in ob(\mathcal{C})$, set $X^{**} = (X^*)^*$, define

$\underline{\psi_X} =$  $X \to X^{**}$.

Lem. $\forall X \in ob(\mathcal{C})$, ψ_X is invertible set.

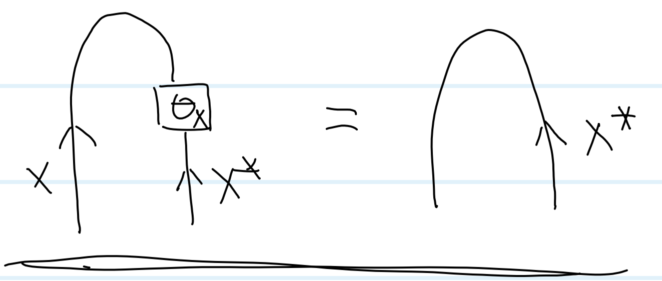
$\phi_X =$ , $\theta_X =$ , $\nu_X =$ 

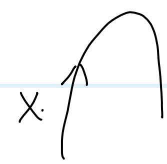
Then $\psi_X = \phi_X$, $\psi_X^{-1} = \theta_X = \nu_X$.

pf: $\theta_X \psi_X = id_X = \nu_X \phi_X$, $\psi_X \theta_X = id_{X^{**}} = \phi_X \nu_X$.

$\Rightarrow \psi_X^{-1} = \theta_X$, $\phi_X^{-1} = \nu_X$, Need to show that $\theta_X = \nu_X$.

$\Leftarrow$ show that

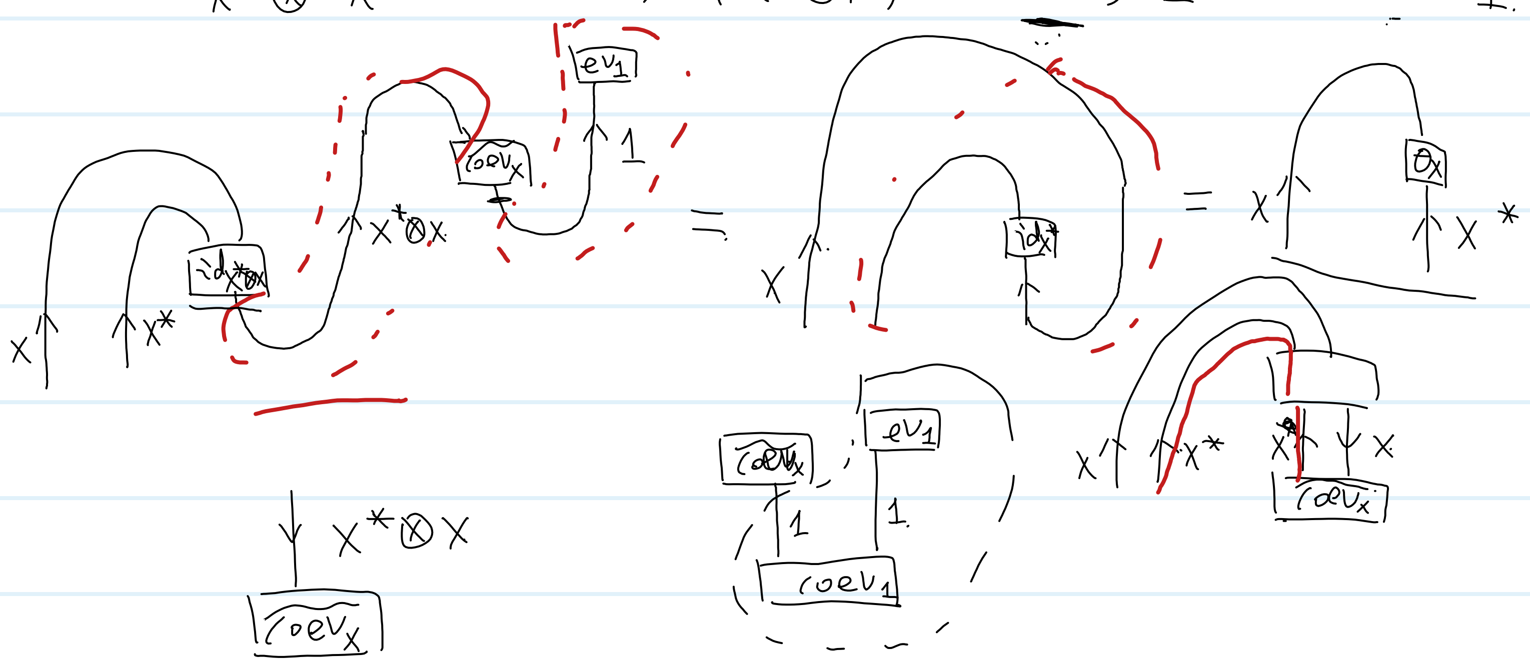


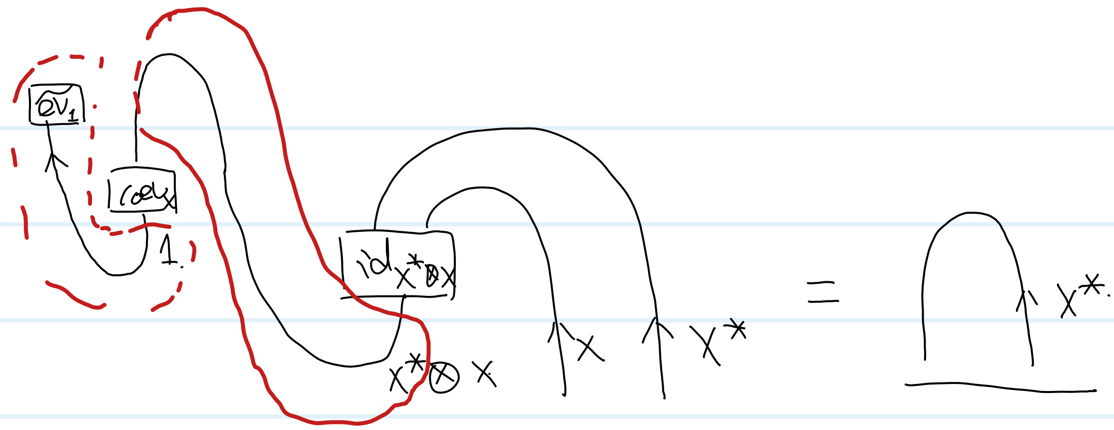


$X^* \otimes X^{**} \to 1$.

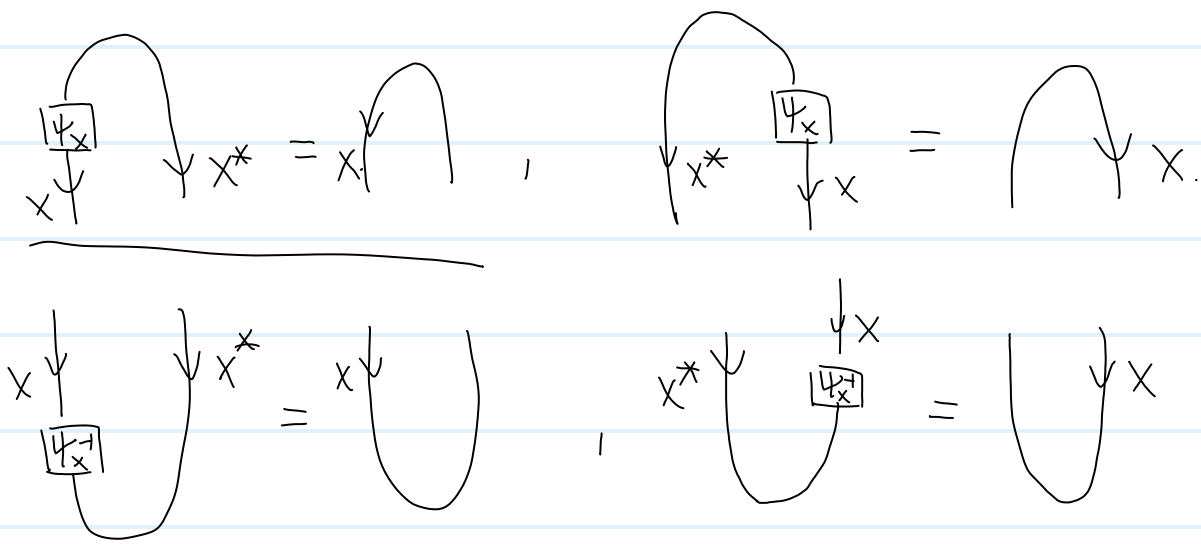
Consider

$$X^* \otimes X^{**} \xrightarrow{?^*_2(X, X^*)} (X^* \otimes X)^* \xrightarrow{(\widetilde{coev}_X)^*} 1^* \xrightarrow{ev_1 = \widetilde{ev}_1} 1.$$





Lemma. $\forall\ X \in ob(\mathcal{C})$.



□