Def. category $\mathcal{C}$

(1). $\underline{Ob(\mathcal{C})}$ (2) $\forall X, Y \in Ob(\mathcal{C})$, $Hom_{\mathcal{C}}(X,Y)$. morphism

(3). $X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow \circ$. $g \circ f : X \to Z$.

$(h \circ g) \circ f = h \circ (g \circ f)$ $g \circ f = gf$.

$\exists\ id_X \in Hom_{\mathcal{C}}(X,X)$, $id_X \circ f = f$, $f \circ id_X = f$.

Sets.

$f : X \to Y$, $\exists\ g : Y \to X$ $gf = id_X$, $fg = id_Y$. $g = f^{-1}$, $X \simeq Y$.

Funtor : $F : \mathcal{C} \to \mathcal{D}$.

(1) $\forall X \in Ob(\mathcal{C})$, $F(X) \in Ob(\mathcal{D})$.

(2). $F(fg) = F(f)F(g)$, $F(id_X) = id_{F(X)}$.

Natural transformation: Given two funtors $F, G : \mathcal{C} \to \mathcal{D}$.

$f : A \to B$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\circlearrowleft"] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[d, "\eta_A"'] \\ G(A) \end{tikzcd} \qquad \forall A \in Ob(\mathcal{C})$$

natural isomorphism.

$F : \mathcal{C} \to \mathcal{D}$. $G : \mathcal{D} \to \mathcal{C}$. $GF \simeq id_{\mathcal{C}}$, $FG \simeq id_{\mathcal{D}}$.

equivalent. categories.

$\mathcal{C}, \mathcal{D}$, $\mathcal{C} \times \mathcal{D}$ $Ob(\mathcal{C} \times \mathcal{D}) = Ob(\mathcal{C}) \times Ob(\mathcal{D})$.

$Hom_{\mathcal{C} \times \mathcal{D}}((A,B),(X,Y)) = Hom_{\mathcal{C}}(A,X) \times Hom_{\mathcal{D}}(B,Y)$.

$(\alpha, \beta) \circ (f, g) \triangleq (\alpha \circ f, \beta \circ g)$.

Def: Monoidal category $(\mathcal{C}, 1, \otimes, a, l, r)$.

- $\mathcal{C}$ category.
- $1 \in ob(\mathcal{C})$
- $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$.
- $a : (- \otimes -) \otimes - \longrightarrow - \otimes (- \otimes -)$ natural isomorphism.

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "(f \otimes g) \otimes h"] \arrow[d, "a_{A,B,C}"'] \arrow[dr, phantom, "\circlearrowleft"] & (X \otimes Y) \otimes Z \arrow[d, "a_{X,Y,Z}"] \\
A \otimes (B \otimes C) \arrow[r, "f \otimes (g \otimes h)"] & X \otimes (Y \otimes Z).
\end{tikzcd}$$

$f : A \to X$

$g : B \to Y$

$h : C \to Z$

$$(- \otimes -) \otimes - : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$
$$(A, B, C) \longmapsto (A \otimes B) \otimes C$$
$$- \otimes (- \otimes -) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$
$$(A, B, C) \longmapsto A \otimes (B \otimes C).$$

- $l : 1 \otimes - \longrightarrow id_{\mathcal{C}}$ and $r : - \otimes 1 \longrightarrow id_{\mathcal{C}}$ are natural isomorphisms.

$$1 \otimes - : \mathcal{C} \to \mathcal{C}.$$
$$A \longmapsto 1 \otimes A.$$
$$f \longmapsto id_1 \otimes f.$$

$\forall f : A \to B.$

r_A.

$$\begin{tikzcd}
1 \otimes A \arrow[r, "id_1 \otimes f"] \arrow[d, "l_A"'] \arrow[dr, phantom, "\circlearrowleft"] & 1 \otimes B \arrow[d, "l_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Pentagon axiom and triangle axiom.

Def: $\mathcal{C} = (\mathcal{C}, \otimes, 1, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes', 1', a', l', r')$ monoidal categories.

Monoidal functor. $F : \mathcal{C} \to \mathcal{D}$.

$$F_0 : 1' \to F(1).$$

$$F_2(X, Y) : F(X) \otimes' F(Y) \longrightarrow F(X \otimes Y), \quad \forall X, Y \in ob(\mathcal{C}).$$

$\forall X, Y, Z \in ob(\mathcal{C})$

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[rr, "\alpha'_{F(X),F(Y),F(Z)}"] \arrow[d, "F_2(X,Y)\otimes' id_{F(Z)}"'] & & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "id_{F(X)} \otimes' F_2(Y,Z)"] \\
F(X\otimes Y) \otimes' F(Z) \arrow[d, "F_2(X\otimes Y, Z)"'] & \circlearrowright & F(X) \otimes' F(Y \otimes Z) \arrow[d, "F_2(X, Y\otimes Z)"] \\
F((X\otimes Y)\otimes Z) \arrow[rr, "F(\alpha_{X,Y,Z})"] & & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
1' \otimes' F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "F_0 \otimes' id_{F(X)}"'] & F(X) \\
F(1) \otimes' F(X) \arrow[r, "F_2(1,X)"] \arrow[ur, phantom, "\circlearrowright"] & F(1\otimes X) \arrow[u, "F(l_X)"']
\end{tikzcd}
\qquad F(X)\otimes' 1'.$$

F is strong if F_0 and $F_2(X,Y)$ are isomorphisms for any $X, Y \in ob(\mathcal{C})$ $\square$

Def. $F, G : \mathcal{C} \to \mathcal{D}$. monoidal functors.

$\varphi : F \to G$. natural transformation is monoidal if.

$$\varphi_1 F_0 = G_0 \quad \text{and} \quad \varphi_{X\otimes Y} F_2(X,Y) = G_2(X,Y)(\varphi_X \otimes \varphi_Y) \quad \forall X, Y \in ob(\mathcal{C}).$$

Thm. $\mathcal{C}$. $\forall A_1, \cdots A_n \in ob(\mathcal{C})$.

P, Q. two paranthesized tensor products of $A_1, \cdots, A_n$.

possibly with copies of the unit object 1.

$f, g : P \to Q$. obtained by composing tensor products of id_X, a, l, r.

and their inverses.

$\Rightarrow f = g$.

$A_1 \xrightarrow[f]{id_{A_1}} A_1$. $\quad f = id_{A_1}$

Example.

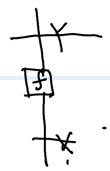
$$\begin{tikzcd} & X \otimes Y & \\ 1 \otimes (X \otimes Y) \arrow[ur, "l_{X \otimes Y}"] & & (1 \otimes X) \otimes Y \arrow[ll, "a_{1,X,Y}"] \arrow[ul, "l_X \otimes id_Y"'] \end{tikzcd}$$

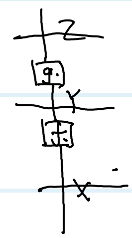
$$\begin{tikzcd} 1 \otimes 1 \arrow[r, bend left, "l_1"] \arrow[r, bend right, "r_1"'] \arrow[r, phantom, "?"] & 1 \end{tikzcd} \Rightarrow l_1 = r_1$$

X, Y

Penrose diagram.

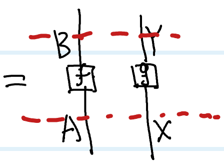
$id_X =$ 

$f: X \to Y =$ 

$gf: X \xrightarrow{f} Y \xrightarrow{g} Z$ 

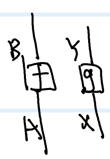
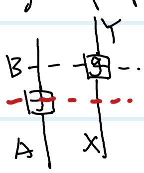
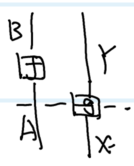
$X \xrightarrow{f} Y$

$f: A \to B$
$g: X \to Y$

$f \otimes g =$ 

$A \otimes X \xrightarrow{f \otimes g} B \otimes Y$.

The level-exchange property.

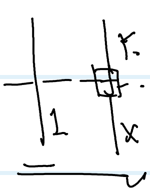
 $=$  $=$ 

$f \otimes g$

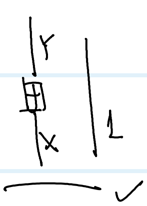
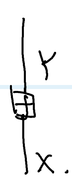
$(id_B \otimes g)(f \otimes id_X)$
$= f \otimes g$.

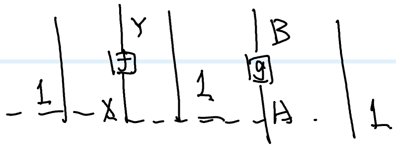
$(f \otimes id_Y)(id_A \otimes g) = f \otimes g$.

l, a, r id_X

 $=$ $\begin{tikzcd} X \arrow[r, "l_X^{-1}"] \arrow[rrr, bend right, "f"'] & 1 \otimes X \arrow[r, "id_1 \otimes f"] & 1 \otimes Y \arrow[r, "l_Y"] & Y \end{tikzcd}$ $=$ 

$$\begin{tikzcd} 1 \otimes X \arrow[r, "id_1 \otimes f"] \arrow[d, "l_X"'] \arrow[dr, phantom, "?"] & 1 \otimes Y \arrow[d, "l_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

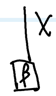
 $=$ $\begin{tikzcd} X \arrow[r] \arrow[rrr, bend right, "f"'] & X \otimes 1 \arrow[r, "f \otimes id_1"] & Y \otimes 1 \arrow[r] & Y \end{tikzcd}$ $=$ 



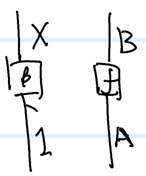
$$1 \otimes X \otimes 1 \otimes A \longrightarrow 1 \otimes Y \otimes 1 \otimes B \to Y \otimes B$$

$\downarrow$ a, l, r, id_X inverse

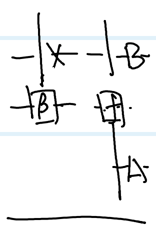
$1 X \otimes A$

$\alpha : 1 \to 1 =$  , $\beta : 1 \to X =$  , $\gamma : X \to 1 =$ 

Lem: For any $\alpha : X \to 1$



$= A \to 1 \otimes A \xrightarrow{\beta \otimes f} X \otimes B.$



$= \underline{A \xrightarrow{\beta \otimes f} X \otimes B.}$ $\qquad 1 \otimes A.$

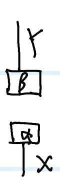
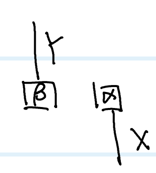
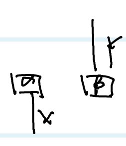
$= A \to 1 \otimes A \xrightarrow{\beta \otimes f} X \otimes B.$

$\Omega : 1 \to Y \otimes X. \qquad w : X \otimes Y \to 1 \qquad (Y \otimes X) \otimes Y$

$Y \xrightarrow{\Omega \otimes id_Y} \boxed{Y \otimes X \otimes Y} \xrightarrow{id_Y \otimes w} Y.$

$Y \to 1 \otimes Y \xrightarrow{\Omega \otimes id_Y} \underline{(Y \otimes X) \otimes Y} \to \underline{Y \otimes (X \otimes Y)} \xrightarrow{id_Y \otimes w} Y \otimes 1 \to Y.$

Lem: $\alpha : X \to 1 \qquad \beta : 1 \to Y.$

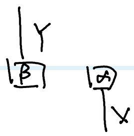
$\beta\alpha =$  $=$  $=$  .

$l_1 = r_1$

Pf:

$$\begin{tikzcd} 1\otimes X \arrow[r, "id_1\otimes\alpha"] \arrow[d, "l_X"'] \arrow[dr, phantom, "\circlearrowleft"] & 1\otimes 1 \arrow[r, "\beta\otimes id_1"] \arrow[d, "l_1 \,|\, r_1"] \arrow[dr, phantom, "\circlearrowleft"] & Y\otimes 1 \arrow[d, "r_Y"] \\ X \arrow[r, "\alpha"] & 1 \arrow[r, "\beta"] & Y \end{tikzcd}$$

$\Rightarrow \beta\alpha = \underline{r_Y(\beta\otimes\alpha)l_X^{-1}}$.



$= X \xrightarrow{\beta\otimes\alpha} Y$

$= X \xrightarrow{l_X^{-1}} 1\otimes X \xrightarrow{\beta\otimes\alpha} Y\otimes 1 \xrightarrow{r_Y} Y.$ $\square$

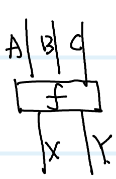
Lem. $End_e(1)$ is commutative.

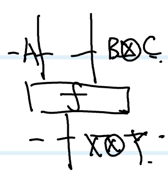
pf: $\forall\ \alpha,\beta \in End_e(1)$.

$$\beta\alpha = \begin{matrix}\boxed{\beta}\\ \boxed{\alpha}\end{matrix} = \boxed{\beta}\ \boxed{\alpha} = \begin{matrix} & \boxed{\alpha} \\ \boxed{\beta} & \end{matrix} = \begin{matrix}\boxed{\alpha}\\ \boxed{\beta}\end{matrix} = \alpha\beta.$$ $\square$

$X_1 \otimes \cdots \otimes X_n \overset{\triangle}{=} (\cdots((X_1\otimes X_2)\otimes X_3)\cdots)\otimes X_n$

$f: X\otimes Y \to \underline{A\otimes B\otimes C}$.

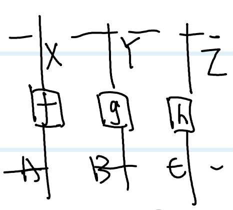




$X\otimes Y \xrightarrow{f} A\otimes B\otimes C$.

$\underline{A\otimes(B\otimes C)} \neq (A\otimes B)\otimes C$.

$A\otimes(B\otimes C) \xrightarrow{\sim} (A\otimes B)\otimes C$



$\underline{A\otimes B}\otimes C \xrightarrow{f\otimes g\otimes h} X\otimes Y\otimes Z$.

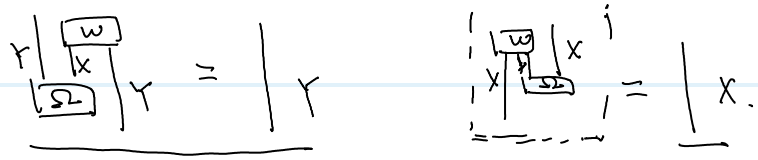
$A, B, C. \qquad X, Y, Z$

$\underline{A\otimes(B\otimes C)} \xrightarrow{f\otimes g\otimes h} X\otimes Y\otimes Z$.

$$\underline{A\otimes(B\otimes C)} \xrightarrow{\circlearrowleft} \underline{A\otimes B\otimes C \xrightarrow{f\otimes g\otimes h} X\otimes Y\otimes Z} \xrightarrow{\circlearrowright} \underline{X\otimes(Y\otimes Z)}$$

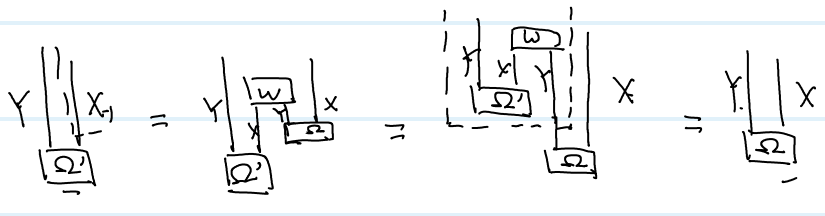
Important!!! subdiagram $\longrightarrow$ total diagram.

Example. If



$\Omega' : 1 \to Y\otimes X$. Then $\Omega = \Omega'$.

Pf:



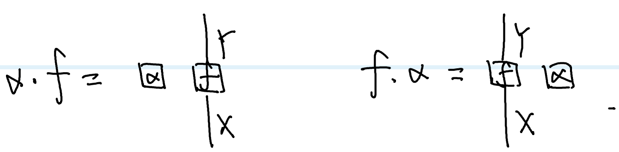
$\square$

$End_{\mathcal{C}}(1)$ acts $Hom_{\mathcal{C}}(X, Y)$, $\forall \alpha \in End_{\mathcal{C}}(1)$, $f \in Hom_{\mathcal{C}}(X, Y)$.

$$\alpha\cdot f = X \to 1\otimes X \xrightarrow{\alpha\otimes f} 1\otimes Y \to Y.$$

$$f\cdot\alpha = Y \to X\otimes 1 \xrightarrow{f\otimes\alpha} Y\otimes 1 \to Y$$

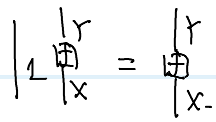
Use diagram.



Lem. $\forall f \in Hom_{\mathcal{C}}(X, Y)$, $\underline{\alpha, \beta} \in End_{\mathcal{C}}(1)$.

$\underline{(\alpha\cdot\beta)\cdot f = (\alpha\beta)\cdot f}$ $\qquad f\cdot(\alpha\beta) = (f\cdot\alpha)\cdot\beta.$ $\qquad \underline{(\alpha\cdot f)\cdot\beta = \alpha\cdot(f\cdot\beta)}.$

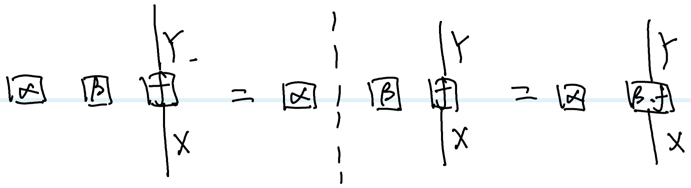
$\underline{id_1 \cdot f} = f = f\cdot id_1.$

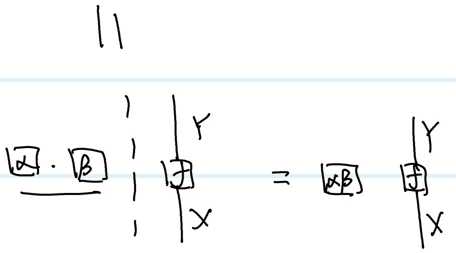


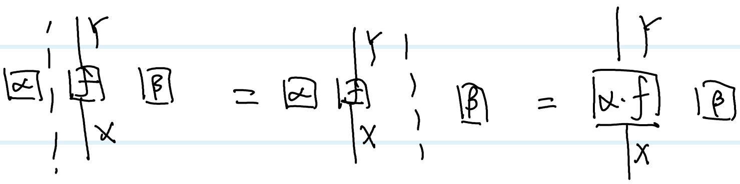
$\underline{\alpha\cdot\beta = \alpha\beta}.$

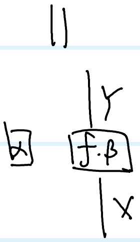
Pf: $\alpha\cdot\beta$ =

$= \alpha\beta.$


$= \alpha \cdot (\beta \cdot f)$


$= (\alpha\beta) \cdot f$.


$= (\alpha \cdot f) \cdot \beta$.


$= \alpha \cdot (f \cdot \beta)$.

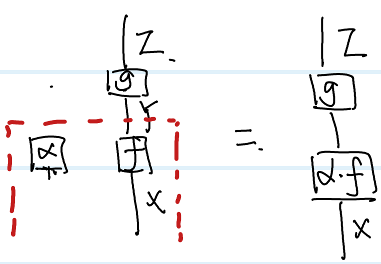
$\square$

Lem. $\forall \alpha \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$.

$$\alpha \cdot (gf) = (\alpha \cdot g) f = g (\alpha \cdot f),$$

$$(gf) \cdot \alpha = (g \cdot \alpha) f = g (f \cdot \alpha),$$

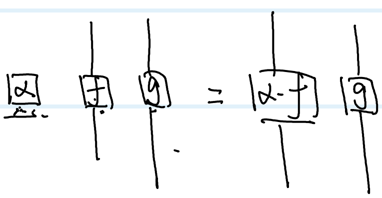
pf: $\alpha \cdot (gf) =$


$= g(\alpha \cdot f)$

$\square$

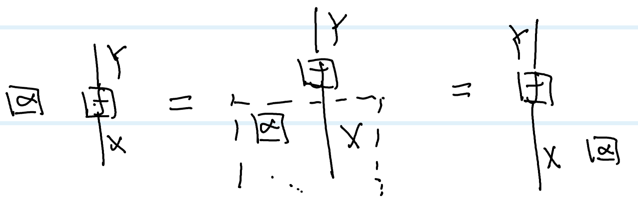
Lem. $\alpha \cdot (f \otimes g) = (\alpha \cdot f) \otimes g \qquad (f \otimes g) \cdot \alpha = f \otimes (g \cdot \alpha)$.

pf. $\alpha \cdot (f \otimes g) =$


$= (\alpha \cdot f) \otimes g$.

Def: $\mathcal{C}$ is pure if $\alpha \cdot f = f \cdot \alpha$ $\forall \alpha \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$, f in $\mathcal{C}$.

$\alpha \cdot \mathrm{id}_X = \mathrm{id}_X \cdot \alpha$.

$\alpha \cdot f =$  $= f \cdot \alpha$.