

CLASSIFICATION OF STRONGLY RATIONAL VERTEX OPERATOR ALGEBRAS WITH CENTRAL AND EFFECTIVE CENTRAL CHARGE ONE

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ABSTRACT. We classify the strongly rational vertex operator algebras whose central charge and effective central charge are both one. Every such algebra is isomorphic to one of

$$V_L, \quad V_L^+, \quad V_{A_1}^{A_4}, \quad V_{A_1}^{S_4}, \quad V_{A_1}^{A_5},$$

where L ranges over the positive-definite even lattices of rank one and $A_1 = \sqrt{2}\mathbb{Z}$.

1. INTRODUCTION

The classification of rational vertex operator algebras at central charge one is a basic rank-one problem in the structure theory of vertex operator algebras. The expected list is classical: positive-definite rank-one lattice vertex operator algebras, their $\mathbb{Z}_2$ -orbifolds, and the three polyhedral orbifolds of the root-lattice vertex operator algebra. Earlier characterizations of the individual models often assumed that the ambient vertex operator algebra decomposes as a sum of highest-weight modules for the Virasoro algebra $L(1, 0)$; see, for example, [9, 10, 8]. This is automatic in the unitary setting, but it does not follow formally from rationality of the ambient vertex operator algebra.

Our purpose is to remove that Virasoro complete-reducibility hypothesis. We call a vertex operator algebra *strongly rational* if it is simple, rational, C_2 -cofinite, self-contragredient, and of CFT type. For a rational vertex operator algebra V , we write $\tilde{c}$ for its effective central charge.

Let

$$A_1 = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = 2,$$

and identify

$$V_{A_1} \cong L_{\widehat{\mathfrak{sl}_2}}(1, 0), \quad \text{Aut}(V_{A_1}) \cong \text{PSL}_2(\mathbb{C}) \cong \text{SO}_3(\mathbb{C}).$$

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We regard A_4, S_4, A_5 as the tetrahedral, octahedral and icosahedral rotation subgroups of $\mathrm{SO}_3(\mathbb{C})$.

Our main theorem is the following.

Theorem 1.1. *Let V be a simple, rational, C_2 -cofinite, self-contragredient vertex operator algebra of CFT type. Assume*

$$c(V) = \tilde{c}(V) = 1.$$

Then there exists a positive-definite even lattice L of rank one such that V is isomorphic to one of

$$V_L, \quad V_L^+, \quad V_{A_1}^{A_4}, \quad V_{A_1}^{S_4}, \quad V_{A_1}^{A_5}.$$

In particular, no assumption that V is completely reducible as an $L(1, 0)$ -module, or that V is a direct sum of Virasoro highest-weight modules, is needed.

There are two parts to the proof. Dong and Jiang's modularity theorem, together with the classical completeness result at $c = 1$, restricts the vacuum character to the five families in Theorem 1.1; see Theorem 2.2. We then prove that each of these characters determines the corresponding vertex operator algebra. The lattice character is handled by Dong–Mason's effective-central-charge theorem. For the remaining four characters, we work only with semisimple subcores of the ambient $L(1, 0)$ -module.

The mechanism is as follows. Set

$$X_m := L(1, m^2), \quad m \in \mathbb{Z}_{\geq 0}.$$

The finite direct sums of the X_m form a symmetric tensor category

$$\mathcal{C}_{\mathrm{sq}} \simeq_{\mathrm{sym}} \mathrm{Rep} \, \mathrm{PSL}_2(\mathbb{C}),$$

with X_m corresponding to the $(2m + 1)$ -dimensional irreducible representation W_{2m} . This equivalence includes the associativity and braiding, not only the fusion rules. Its normalization comes from the compact orbifold

$$V_{A_1}^{SO(3)} = L(1, 0),$$

so the associativity and braiding are the ordinary symmetric ones on $\mathrm{Rep} \, \mathrm{PSL}_2(\mathbb{C})$.

Suppose that square simple Virasoro modules are embedded in a vertex operator algebra V . A theorem of Creutzig–McRae–Yang implies that the coefficient image of an intertwining

operator between two C_1 -cofinite modules is again grading restricted and C_1 -cofinite. The $P(z)$ -tensor-product universal property realizes this image as a quotient of

$$X_a \boxtimes X_b \cong \bigoplus_{j=|a-b|}^{a+b} X_j.$$

The image is consequently semisimple and belongs to the square subcategory. Iteration produces the required semisimple core even when $V \downarrow L(1, 0)$ is not semisimple.

For the three polyhedral characters, the first deviations from the vacuum character occur at conformal weights 9, 16, 36. The invariant bilinear form produces a primary vector at that weight, and a pairing argument at the first singular level shows that its cyclic Virasoro module is X_3, X_4, X_6 , respectively. The square core generated by this module corresponds to a finitely generated commutative $\mathrm{PSL}_2(\mathbb{C})$ -algebra B . Since $B^{\mathrm{PSL}_2(\mathbb{C})} = \mathbb{C}$, affine GIT supplies a unique closed orbit G/H and a quotient

$$B \twoheadrightarrow \mathcal{O}(G/H).$$

The first invariant degree identifies H as A_4, S_4, A_5 . Comparison with the prescribed graded character then shows that the quotient is an isomorphism.

The V_L^+ branch uses a different reconstruction. Its square core is generated by a weight-four primary, corresponding under $\mathrm{Rep} \mathrm{PSL}_2(\mathbb{C})$ to

$$W_4 \cong \mathrm{Sym}_0^2(\mathbb{C}^3).$$

When the weight-four primary space is one-dimensional, the invariant quadratic form and the unique quadratic covariant yield scalar relations. Their common zero scheme is the homogeneous orbit $G/N(T)$, and hence

$$\langle L(1, 0), X_2 \rangle \cong M(1)^+.$$

The first lattice primary is then extracted orthogonally as an irreducible $M(1)^+$ -module $M(1, \lambda)$. The tensor-category theorem of Adamović, Lin, and Yang supplies the ambient braided category, while the genuine compact $O(2)$ -action on the rank-one lattice VOA $V_{\mathbb{Z}\lambda}$ fixes the symmetric normalization of the relevant integral-charge subcategory. This subcategory is equivalent to

$$\mathrm{Rep} O_2(\mathbb{C}).$$

The extension is thus controlled by a commutative O_2 -algebra generated by the standard two-dimensional representation. The first lattice primary has nonzero norm, so the invariant

quadratic polynomial is specialized to a nonzero constant. The resulting quadric is a single O_2 -orbit and reconstructs V_L^+ .

The norms 2, 4, 6, 8 require separate arguments because the first lattice primary overlaps the universal low-weight structure. The first two cases follow from the Dong–Mason and Zhang–Dong uniqueness theorems. At norm 6, dimension saturation at weight nine shows that the weight-four primary generates X_2 . At norm 8, the weight-four primary space is two-dimensional. The map that sends a primary to the image of the level-five singular vector is shown to vanish by the nondegeneracy of the invariant form. A regular primary is then selected by its quadratic and cubic invariants; its orbit is G/D_2 . The lattice calculation

$$V_{A_1}^{D_2} = V_{2A_1}^+$$

finishes the last case.

The paper is organized as follows. Section 2 records the character reduction, the square tensor category, the direct-limit extension input, and the homogeneous-coordinate-algebra dictionary. Section 3 develops the first-primary and first-singular arguments and proves the exceptional character rigidity theorem. Section 4 reconstructs the $M(1)^+$ core. Section 5 establishes the O_2 charge category and proves the rank-one orbifold rigidity for lattice norm > 8 . Section 6 treats the four low-norm cases. Section 7 proves the lattice branch and combines all character branches to prove Theorem 1.1.

2. PRELIMINARIES AND CATEGORICAL INPUT

2.1. The classical character reduction. For a rational vertex operator algebra V , let $\lambda_{\min}$ be the least conformal weight among its irreducible modules and recall

$$\tilde{c}(V) = c(V) - 24\lambda_{\min}.$$

The hypothesis $c = \tilde{c} = 1$ is essential in the character classification used below.

Theorem 2.1 (Dong–Jiang). *Let V be simple, rational, C_2 -cofinite and self-contragredient. Let $M^1, \dots, M^r$ be the irreducible ordinary V -modules and write*

$$Z_i(\tau) = \text{tr}_{M^i} q^{L(0) - c/24}.$$

Then every Z_i is a modular function for a congruence subgroup and

$$\sum_{i=1}^r |Z_i(\tau)|^2$$

is $SL_2(\mathbb{Z})$ -invariant.

Proof. This is [8, Theorem 2.6]. Its hypotheses are rationality, C_2 -cofiniteness, self-duality, and simplicity. In the terminology of that paper, self-duality means $V \cong V'$ as a V -module, which is precisely self-contragredience. The theorem contains no assumption on the restriction to the Virasoro algebra. $\square$

Theorem 2.2 (Classical $c = 1$ vacuum-character reduction). *Let V satisfy the hypotheses of Theorem 1.1. Then there is a positive-definite even rank-one lattice L such that*

$$\text{ch } V \in \{ \text{ch } V_L, \text{ch } V_L^+, \text{ch } V_{A_1}^{A_4}, \text{ch } V_{A_1}^{S_4}, \text{ch } V_{A_1}^{A_5} \}.$$

Proof. Theorem 2.1 supplies exactly the two modularity assumptions in [8, Theorem 2.9]. Since V is of CFT type and $c = \tilde{c} = 1$, that theorem gives the five vacuum characters displayed above. Its character-theoretic input is the classical $c = 1$ completeness result of Kiritsis [13]. As Dong and Jiang note immediately after Theorem 2.9, their Theorem 2.6 supplies the required modularity; no decomposition into Virasoro highest-weight modules enters this step. $\square$

2.2. The square Virasoro category. Let $L(1, h)$ denote the irreducible highest-weight module for the Virasoro vertex operator algebra $L(1, 0)$ of central charge one and highest weight h . Set

$$X_m := L(1, m^2), \quad m \in \mathbb{Z}_{\geq 0}.$$

We use W_{2m} for the $(2m + 1)$ -dimensional irreducible algebraic representation of $\text{PSL}_2(\mathbb{C})$.

Theorem 2.3 (Compact-orbifold normalization of the square category). *Let $G = \text{PSL}_2(\mathbb{C}) \cong \text{SO}_3(\mathbb{C})$ and let $K = \text{SO}(3)$ be a compact real form. Then*

$$V_{A_1}^K = L(1, 0), \quad V_{A_1} \cong \bigoplus_{m \geq 0} W_{2m} \otimes X_m$$

as a $K \times L(1, 0)$ -module. The full subcategory $\mathcal{C}_{\text{sq}}$ consisting of finite direct sums of the X_m is a symmetric tensor category and there is a symmetric tensor equivalence

$$\Phi : \text{Rep } G \xrightarrow{\sim} \mathcal{C}_{\text{sq}}, \quad W_{2m} \mapsto X_m.$$

The equivalence may be equipped with $P(z)$ -tensor constraints, for all $z \in \mathbb{C}^\times$, which are compatible with parallel transport, associativity and braiding. In particular, it is a symmetric vertex tensor equivalence in the sense used below. In particular,

$$X_a \boxtimes X_b \cong \bigoplus_{j=|a-b|}^{a+b} X_j.$$

Proof. McRae’s compact-orbifold theorem identifies the tensor category generated by the multiplicity spaces in a compact-group orbifold with a cocycle modification of the representation category of the compact group [19]. Example 4.12 of that paper treats the weight-lattice abelian intertwining algebra and, within it, the ordinary root-lattice VOA V_{A_1} . It identifies $V_{A_1}^{SO(3)}$ with $L(1, 0)$ and shows that the smaller subcategory coming from V_{A_1} —the one indexed by the even $SU(2)$ weights—is symmetric tensor equivalent to $\text{Rep } SO(3)$. These are precisely the modules $L(1, m^2)$. Corollary 4.8 supplies the tensor equivalence, while Remarks 4.9–4.10 give independence from the ambient module category and the asserted $P(z)$ -compatibility. Finally, finite-dimensional continuous representations of $SO(3)$ identify with algebraic representations of its complexification $\text{PSL}_2(\mathbb{C})$. The last formula is the Clebsch–Gordan rule. $\square$

Remark 2.4. *The distinction between the category in Theorem 2.3 and the larger $SU(2)$ -type category at $c = 1$ is important. The latter may be described with an abelian 3-cocycle twist. We never need to trivialize that larger twist: the square subcategory is normalized directly by the ordinary VOA V_{A_1} and the compact $SO(3)$ -orbifold.*

2.3. The full C_1 -cofinite Virasoro category and direct limits. At central charge one, the universal and simple Virasoro vertex operator algebras coincide. Indeed, the universal algebra denoted $M(c, 0)$ in [4] is simple unless $c = c_{p,q}$ for coprime integers $p, q \geq 2$ [4, Corollary 2.2.5(4)]. Since

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq},$$

the equality $c_{p,q} = 1$ would force $p = q$, contrary to the coprimality assumption. Thus $M(1, 0) = L(1, 0)$ in the notation of [4].

Let $\mathcal{O}_1$ be the category of lower-bounded C_1 -cofinite generalized $L(1, 0)$ -modules. By [4, Theorem 3.1.4], this is the finite-length category considered there; in particular, its objects are grading restricted and it is abelian. By [4, Theorem 4.2.6], it carries the required vertex and braided tensor category structure. The argument below uses only its tensor-product universal property and the semisimplicity of the square subcategory.

The proofs below use a direct-limit completion. An intertwining operator may initially take values in a large ind-object, or even in an ambient vertex operator algebra not known a priori to lie in the finite-length category; this is the point that requires care.

Lemma 2.5 (The square category inside $\mathcal{O}_1$). *For every $m \geq 0$, the module $X_m = L(1, m^2)$ is C_1 -cofinite. Consequently $\mathcal{C}_{\text{sq}}$ is a full semisimple subcategory of $\mathcal{O}_1$. For $A, B \in \mathcal{C}_{\text{sq}}$, their $P(z)$ -tensor product in $\mathcal{C}_{\text{sq}}$ is canonically the $P(z)$ -tensor product computed in $\mathcal{O}_1$.*

Proof. In the notation of [4], central charge one corresponds to $t = 1$, and

$$h_{r,s}(1) = \frac{(r-s)^2}{4}.$$

Thus

$$m^2 = h_{2m+1,1}(1).$$

Corollary 2.2.7(1) of [4] now shows that X_m is C_1 -cofinite. Finite direct sums of the X_m therefore belong to $\mathcal{O}_1$.

The $P(z)$ -constraints in Theorem 2.3 are the ones constructed from the ambient Virasoro intertwining operators. By [19, Corollary 4.8 and Remarks 4.9–4.10], these constraints are isomorphisms for every $z \neq 0$, and the resulting tensor products are independent of the ambient Virasoro module category. Hence, after viewing $\mathcal{C}_{\text{sq}}$ as a full subcategory of $\mathcal{O}_1$, its $P(z)$ -tensor products are the ambient ones. $\square$

Definition 2.6. *Let A, B be $L(1, 0)$ -modules and X a generalized module. For an intertwining operator*

$$\mathcal{Y} : A \otimes B \longrightarrow X\{z\}[\log z],$$

let $\text{CoeffIm}(\mathcal{Y})$ be the smallest Virasoro submodule of X containing all coefficients of all series $\mathcal{Y}(a, z)b$.

Lemma 2.7 (Coefficient images). *Let $A, B \in \mathcal{C}_{\text{sq}}$ and let X be any generalized $L(1, 0)$ -module. If $\mathcal{Y}$ is an intertwining operator of type $\begin{pmatrix} X \\ A \ B \end{pmatrix}$, then*

$$I := \text{CoeffIm}(\mathcal{Y})$$

is a grading-restricted C_1 -cofinite module. In particular, $I \in \mathcal{O}_1$, and the $P(z)$ -tensor-product universal property yields an epimorphism

$$A \boxtimes_{\mathcal{O}_1} B \twoheadrightarrow I.$$

Consequently, $I \in \mathcal{C}_{\text{sq}}$.

Proof. The Jacobi identity shows that the span of the coefficients of $\mathcal{Y}(a, z)b$ is already a Virasoro submodule of X ; thus it is I . The same series therefore defines a corestricted intertwining operator whose coefficients span its target:

$$\mathcal{Y}^I : A \otimes B \longrightarrow I\{z\}[\log z].$$

The Jacobi identity, the $L(-1)$ -derivative property, and the grading conditions are unchanged under this corestriction, because $I \hookrightarrow X$ is a module homomorphism. By Lemma 2.5, the two inputs are C_1 -cofinite. Corollary 2.14 of [6], whose target may be an arbitrary generalized module, now gives $I \in \mathcal{O}_1$. The ambient $P(z)$ -tensor universal property in $\mathcal{O}_1$, applied with target I , gives a morphism

$$f : A \boxtimes_{\mathcal{O}_1} B \longrightarrow I.$$

Let $I_{\mathcal{Y}^I}$ be the $P(z)$ -intertwining map obtained from $\mathcal{Y}^I$ using the chosen branch. By construction,

$$I_{\mathcal{Y}^I} = \bar{f} \circ \boxtimes_{P(z)}.$$

For the chosen branch of $\log z$, the standard bijection between logarithmic intertwining operators and $P(z)$ -intertwining maps is inverted by the $L(0)$ -conjugation formula [19, Equation (2.1)]. Applying this inverse to the displayed identity shows that $\mathcal{Y}^I$ is obtained by applying f to the tensor-product intertwining operator. Since f commutes with the generalized $L(0)$ -weight projections, every formal coefficient of $\mathcal{Y}^I(a, x)b$ lies in $\text{Im } f$. These coefficients span I by definition, and hence $\text{Im } f = I$. Thus f is an epimorphism. Lemma 2.5 identifies the source with the tensor product in $\mathcal{C}_{\text{sq}}$. Theorem 2.3 makes it a finite semisimple direct sum of square simples. Its quotient I is therefore semisimple, with simple constituents among the same X_m ; hence $I \in \mathcal{C}_{\text{sq}}$. $\square$

Lemma 2.8 (Ambient subquotients of the square category). *The full subcategory $\mathcal{C}_{\text{sq}} \subset \mathcal{O}_1$ is closed under ambient $L(1, 0)$ -submodules and quotients, and the inclusion $\mathcal{C}_{\text{sq}} \hookrightarrow \mathcal{O}_1$ is exact.*

Proof. Let

$$M = \bigoplus_{m \in F} X_m^{\oplus n_m} \in \mathcal{C}_{\text{sq}}, \quad F \subset \mathbb{Z}_{\geq 0} \text{ finite.}$$

As an ordinary $L(1, 0)$ -module, M is a finite direct sum of simple modules and is therefore semisimple. Every ambient $L(1, 0)$ -submodule of M is consequently a direct summand and a finite direct sum of simple modules isomorphic to some of the X_m . The same is true of

every ambient quotient. Thus kernels and cokernels of morphisms in the full subcategory agree with their ambient kernels and cokernels, which proves the assertion. $\square$

Theorem 2.9 (Universal image condition and the direct-limit category). *The category $\mathcal{C}_{\text{sq}}$ satisfies the hypotheses of the direct-limit completion theorem of Creutzig–McRae–Yang. In particular, $\text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}})$ has the required vertex and braided tensor category structure. The extension–algebra correspondence applies to its commutative algebra objects that are integrally $L(0)$ -graded and satisfy the grading restrictions in [6, Theorem 1.3].*

Proof. We verify the conditions of [6, Theorem 1.1] in their stated order. The base VOA $X_0 = L(1, 0)$ belongs to $\mathcal{C}_{\text{sq}}$. Lemma 2.8 proves closure under submodules and quotients in the ambient generalized-module category; finite direct sums are immediate. Every object is a finite direct sum of irreducible highest-weight modules and is therefore finitely generated. By [19, Corollary 4.8 and Remarks 4.9–4.10], the tensor equivalence of Theorem 2.3 is compatible with the canonical $P(z)$ -intertwining maps for every $z \in \mathbb{C}^\times$ and with parallel transport. Thus the $P(z)$ -tensor products, associativity, and braiding on $\mathcal{C}_{\text{sq}}$ are the restrictions of those in the ambient Virasoro vertex tensor category. This verifies condition (4).

For condition (5), let $A, B \in \mathcal{C}_{\text{sq}}$, let $X \in \text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}})$, and let $\mathcal{Y}$ be an intertwining operator of type $\left(\begin{smallmatrix} X \\ A \ B \end{smallmatrix}\right)$. By [6, Corollary 2.14], its coefficient image I is grading restricted and C_1 -cofinite. Lemma 2.7 realizes I as a quotient of the finite semisimple ambient tensor product, so $I \in \mathcal{C}_{\text{sq}}$. This verifies the last hypothesis of [6, Theorem 1.1]. That theorem provides the direct-limit tensor structure, and [6, Theorem 1.3] gives the extension statement. $\square$

Lemma 2.10 (Ind-extension of a semisimple tensor equivalence). *Let G be a reductive algebraic group and let $\mathcal{C}$ be a semisimple vertex tensor category whose objects are finite direct sums of its simple objects. Suppose that $\mathcal{C}$ satisfies the hypotheses of [6, Theorem 1.1] and that*

$$\Psi : \text{Rep } G \xrightarrow{\sim} \mathcal{C}$$

is a symmetric vertex tensor equivalence. Then Ψ extends to an exact cocontinuous symmetric monoidal equivalence

$$\widehat{\Psi} : \text{Rep}^{\text{rat}} G \xrightarrow{\sim} \text{Ind}_{\text{CMY}}(\mathcal{C}),$$

where $\text{Rep}^{\text{rat}} G$ is the category of arbitrary rational algebraic G -modules. If $\{S_i\}_{i \in I}$ is a set of simple G -modules and

$$M = \bigoplus_{i \in I} S_i \otimes E_i,$$

then

$$\widehat{\Psi}(M) = \bigoplus_{i \in I} \Psi(S_i) \otimes E_i.$$

Proof. Choose representatives $\{S_\lambda\}_{\lambda \in \Lambda}$ of the simple rational G -modules and put $T_\lambda = \Psi(S_\lambda)$. Since G is linearly reductive over $\mathbb{C}$, every rational G -module is locally finite and semisimple, and the canonical evaluation map

$$\bigoplus_{\lambda \in \Lambda} S_\lambda \otimes \text{Hom}_G(S_\lambda, M) \longrightarrow M$$

is an isomorphism. Thus $\text{Rep}^{\text{rat}} G$ is the ordinary Ind-completion of $\text{Rep } G$.

On the other hand, an object X of $\text{Ind}_{\text{CMY}}(\mathcal{C})$ is the directed union of its finite $\mathcal{C}$ -subobjects. Each of these is semisimple. Hence every vector of X lies in a finite sum of simple $\mathcal{C}$ -subobjects. Any finite linear relation among distinct simple summands is already contained in one finite $\mathcal{C}$ -object, where the usual isotypic decomposition is direct. It follows that the evaluation map

$$\bigoplus_{\lambda \in \Lambda} T_\lambda \otimes \text{Hom}_{\mathcal{C}}(T_\lambda, X) \longrightarrow X$$

is an isomorphism. In particular, $\text{Ind}_{\text{CMY}}(\mathcal{C})$ is the ordinary Ind-completion of $\mathcal{C}$ as an abelian category.

Define $\widehat{\Psi}$ by replacing S_λ by T_λ in the isotypic decomposition and leaving the multiplicity spaces unchanged. Schur's lemma identifies morphisms on both sides with the same families of linear maps on multiplicity spaces. Thus $\widehat{\Psi}$ is fully faithful and essentially surjective. It is exact. It also preserves arbitrary direct sums; since colimits in these cocomplete semisimple abelian categories are obtained from direct sums and cokernels, it is cocontinuous.

By the construction of the CMY tensor product,

$$X \boxtimes Y = \varinjlim_{X_0 \subset X, Y_0 \subset Y} (X_0 \boxtimes Y_0),$$

where X_0, Y_0 range over finite $\mathcal{C}$ -subobjects. The tensor constraints of Ψ on each pair (X_0, Y_0) form a compatible morphism of these directed systems and hence induce an isomorphism

$$\widehat{\Psi}(M) \boxtimes \widehat{\Psi}(N) \xrightarrow{\sim} \widehat{\Psi}(M \otimes N).$$

Every vector occurring in a coherence diagram lies in finite subobjects, on which the diagram is the corresponding coherence diagram for Ψ . The unit, associativity, and symmetry constraints therefore satisfy the required coherence identities. The same finite-subobject

argument, using the vertex tensor compatibility of Ψ , gives compatibility with the $P(z)$ -constraints and parallel transport. Thus $\widehat{\Psi}$ is a strong symmetric monoidal equivalence. $\square$

Lemma 2.11 (Commuting actions and invariants). *In the notation of Lemma 2.10, let H be a reductive algebraic group and let M be a rational $G \times H$ -module. Then the commuting H -action on M induces a rational H -action on $\widehat{\Psi}(M)$ by morphisms in $\text{Ind}_{\text{CMY}}(\mathcal{C})$. If M is a commutative rational G -algebra and H acts by algebra automorphisms, then H acts on $\widehat{\Psi}(M)$ by commutative-algebra-object automorphisms, and*

$$\widehat{\Psi}(M^H) \cong \widehat{\Psi}(M)^H$$

as commutative algebra objects.

Proof. Write the left G -isotypic decomposition as

$$M = \bigoplus_{\lambda} S_{\lambda} \otimes E_{\lambda}.$$

Because the H -action commutes with G , each multiplicity space E_{λ} is a rational H -module, and

$$\widehat{\Psi}(M) = \bigoplus_{\lambda} \Psi(S_{\lambda}) \otimes E_{\lambda}$$

with H acting on the second factors. Since H is linearly reductive,

$$\widehat{\Psi}(M)^H = \bigoplus_{\lambda} \Psi(S_{\lambda}) \otimes E_{\lambda}^H = \widehat{\Psi}(M^H).$$

If M is an algebra, naturality and strong monoidality of $\widehat{\Psi}$ show that the transported action preserves multiplication and unit, and the displayed identification is an algebra-object isomorphism. $\square$

Applying Lemma 2.10 to Theorem 2.3, we henceforth write

$$\widehat{\Phi} : \text{Rep}^{\text{rat}} G \xrightarrow{\sim} \text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}})$$

for the resulting equivalence.

2.4. Admissible commutative algebras and ideals. The algebra-object/extension correspondence will be used for infinite direct sums. We therefore record explicitly the grading and twist conditions that turn an ind-algebra into an honest grading-restricted vertex operator algebra.

Definition 2.12. *Let V_0 be a vertex operator algebra and $\mathcal{C}$ a vertex tensor category of V_0 -modules satisfying the hypotheses of [6, Theorem 1.1]. A commutative algebra object $A \in \text{CAlg}(\text{Ind}_{\text{CMY}}(\mathcal{C}))$ will be called extension-admissible if:*

- (i) the unit $\mathbf{1} \rightarrow A$ is injective and $\text{Hom}(\mathbf{1}, A) = \mathbb{C}$;*
- (ii) the twist $\theta_A = e^{2\pi i L(0)}$ is the identity;*
- (iii) A is lower bounded, $L(0)$ acts semisimply, and every conformal-weight space is finite dimensional.*

For finite algebra objects this is the extension correspondence of Huang–Kirillov–Lepowsky [12] and Creutzig–Kanade–McRae [5]; for the ind-objects used here, it is [6, Theorem 1.3]. Under that correspondence the algebra unit $V_0 \rightarrow A$ is the structural vertex-operator-algebra homomorphism from the base VOA into the reconstructed algebra. Thus the injectivity required in (i) makes the reconstructed VOA a genuine extension of V_0 .

Condition (iii) decomposes A into $L(0)$ -eigenspaces. If h is an eigenvalue, condition (ii) gives $e^{2\pi i h} = 1$, and hence $h \in \mathbb{Z}$. The lower bound and finite-dimensional eigenspaces in (iii) supply the remaining grading restrictions. Thus the theorem produces a grading-restricted, integrally graded VOA extension with the same conformal vector. In the two direct-limit categories used below, every object is an arbitrary direct sum of the specified simple modules. The tensor unit is the unique such simple of lowest weight zero and every other simple has strictly positive lowest conformal weight. Hence $\text{Hom}(\mathbf{1}, A) = \mathbb{C}$ implies $A_0 = \mathbb{C}\mathbf{1}$, while the lower bounds of the simple summands rule out negative weights. The extensions considered here are therefore of CFT type.

Proposition 2.13 (Ideal interface). *Let G be a reductive algebraic group and let $\mathcal{C}$ be a full abelian vertex tensor subcategory of grading-restricted generalized V_0 -modules. Assume that $\mathcal{C}$ satisfies the hypotheses of [6, Theorem 1.1], that its inclusion in the ambient module category is exact and closed under ambient V_0 -submodules, and that a symmetric tensor equivalence*

$$\Phi : \text{Rep } G \xrightarrow{\sim} \mathcal{C}$$

extends to an exact symmetric monoidal equivalence

$$\widehat{\Phi} : \text{Rep}^{\text{rat}} G \xrightarrow{\sim} \text{Ind}_{\text{CMY}}(\mathcal{C}).$$

Let B be a commutative rational G -algebra. Put $A = \widehat{\Phi}(B)$ and assume that A is extension-admissible. Let U be the corresponding VOA extension of V_0 . By a categorical algebra ideal we mean a monomorphism $i : J \hookrightarrow A$ for which there is a morphism $\mu_J : A \boxtimes J \rightarrow J$ satisfying

$$i \circ \mu_J = \mu_A \circ (\text{id}_A \boxtimes i).$$

Then the following are naturally equivalent:

- (i) VOA ideals of U ;*
- (ii) categorical algebra ideals of A ;*
- (iii) rational G -stable ordinary algebra ideals of B .*

Consequently,

$$U \text{ is simple} \iff B \text{ has no nonzero proper } G\text{-stable ideals.}$$

Proof. Let $J \subset U$ be a VOA ideal. If $F_1, F_2 \subset U$ are finite $\mathcal{C}$ -subobjects, then $F_1 + F_2$ is the image of $F_1 \oplus F_2 \rightarrow U$ and belongs to $\mathcal{C}$ by quotient closure. Thus the finite $\mathcal{C}$ -subobjects of U form a directed system. Since $\mathcal{C}$ is closed under ambient submodules, $J \cap F \in \mathcal{C}$ for each F , and

$$J = \varinjlim_F (J \cap F) \in \text{Ind}_{\text{CMY}}(\mathcal{C}).$$

The restricted vertex operator $Y_U|_{U \otimes J}$ is an intertwining operator with target J . The ind-object universal property [6, Theorem 6.2(2)] gives a unique morphism

$$\mu_J : A \boxtimes J \longrightarrow J$$

and, for the inclusion $i : J \hookrightarrow A$, uniqueness gives

$$i \circ \mu_J = \mu_A \circ (\text{id}_A \boxtimes i).$$

Hence J is a categorical algebra ideal.

Conversely, suppose that $i : J \hookrightarrow A$ and μ_J satisfy this identity. Since A is grading restricted and $L(0)$ is semisimple, we identify i with its graded image and regard J as an actual V_0 -submodule of $A = U$. Under the bijection of [6, Theorem 6.2(2)], the algebra-ideal identity is equivalent to

$$i Y_J(a, z)j = Y_U(a, z)i(j).$$

Thus every coefficient of $Y_U(a, z)i(j)$ belongs to $i(J)$, and

$$Y_U(U, z)J \subset J((z)).$$

Because J is a V_0 -submodule, it is $L(-1)$ -stable; skew-symmetry then gives $Y_U(J, z)U \subset J((z))$. Thus J is a VOA ideal.

The equivalence $\widehat{\Phi}$ preserves subobjects and multiplication morphisms, so categorical algebra ideals of A are precisely algebra ideals of B in $\text{Rep}^{\text{rat}} G$. Subobjects of a rational G -module are exactly its G -stable linear subspaces, and such subspaces carry the induced rational action. These are therefore precisely the rational G -stable ordinary algebra ideals of B . $\square$

Proposition 2.14 (Algebra subobjects and vertex subalgebras). *Let A be a commutative algebra object in $\text{Ind}_{\text{CMY}}(\mathcal{C})$ that corresponds under the CMY extension theorem to a grading-restricted VOA extension U of V_0 . Suppose $i : B \hookrightarrow A$ is a unital commutative algebra subobject. Identify its image $C = i(B)$ with an actual V_0 -submodule of U , and assume that C is grading restricted. Then C is a vertex operator subalgebra of U containing V_0 .*

Proof. Let $\mu_B : B \boxtimes B \rightarrow B$ be the restricted multiplication. The algebra-subobject identity

$$i \circ \mu_B = \mu_A \circ (i \boxtimes i)$$

and [6, Theorem 6.2(2)] identify its associated $P(1)$ -map with the restriction of the ambient multiplication of U to $C \otimes C$. Hence this $P(1)$ -map has image in $\overline{C}$. Since C is $L(0)$ -stable and grading restricted, each homogeneous projection lies in C . Applying the inverse $P(1)$ formula [19, Equation (2.1)] therefore gives

$$a_n b \in C \quad (a, b \in C, n \in \mathbb{Z}).$$

The unit subobject is the embedded base VOA V_0 , so C contains the vacuum and conformal vector. Thus C is a vertex operator subalgebra of U . $\square$

Lemma 2.15 (Homogeneous coordinate algebras are G -simple). *Let G/H be a reduced transitive affine G -variety. Then $\mathcal{O}(G/H)$ has no nonzero proper G -stable ideals.*

Proof. Let I be a proper G -stable ideal. Its zero set $V(I)$ is a nonempty G -stable closed subset of the transitive G -variety G/H , and therefore $V(I) = G/H$. Hence $\sqrt{I} = 0$. Since $\mathcal{O}(G/H)$ is reduced, $I = 0$. $\square$

2.5. The regular algebra and the fixed-point dictionary. For $g \in G$ and $f \in \mathcal{O}(G)$, set

$$(\lambda(g)f)(x) = f(g^{-1}x), \quad (\rho(g)f)(x) = f(xg).$$

We use λ for the categorical G -action. The right action ρ commutes with λ and acts by algebra automorphisms. It will become the automorphism action on the reconstructed VOA.

Lemma 2.16 (Extension admissibility for the homogeneous algebras). *Let $H \leq G = \mathrm{PSL}_2(\mathbb{C})$ be reductive and suppose that G/H is affine. Then $\widehat{\Phi}(\mathcal{O}(G/H))$ is extension-admissible. More explicitly,*

$$\mathcal{O}(G/H) \cong \bigoplus_{m \geq 0} W_{2m} \otimes (W_{2m}^*)^H$$

under the left regular action, and therefore

$$\widehat{\Phi}(\mathcal{O}(G/H)) \cong \bigoplus_{m \geq 0} \dim W_{2m}^H X_m.$$

Proof. With the conventions above, algebraic Peter–Weyl gives

$$\mathcal{O}(G) = \bigoplus_{m \geq 0} W_{2m}^* \otimes W_{2m},$$

where the two factors carry the left and right actions, respectively. Thus

$$\mathcal{O}(G/H) = \mathcal{O}(G)^{H_{\text{right}}} = \bigoplus_{m \geq 0} W_{2m}^* \otimes W_{2m}^H.$$

Since W_{2m} is self-dual, this is the decomposition stated in the lemma. The trivial representation occurs with multiplicity one, so $\mathrm{Hom}(\mathbf{1}, A) = \mathbb{C}$. The algebraic unit $\mathbb{C} \rightarrow \mathcal{O}(G/H)$ sends 1 to the nonzero constant function 1; since $\widehat{\Phi}$ is faithful, its image is a nonzero morphism from the simple tensor unit and is therefore a monomorphism. Every $X_m = L(1, m^2)$ has integral conformal spectrum $m^2 + \mathbb{Z}_{\geq 0}$, so the twist is the identity. For a fixed absolute conformal weight N , only $m^2 \leq N$ can contribute. There are finitely many such m , each multiplicity $\dim W_{2m}^H$ is finite, and each homogeneous space in X_m is finite dimensional. Thus the resulting ind-object is lower bounded, $L(0)$ -semisimple and grading restricted. $\square$

Theorem 2.17 (The regular algebra). *Let U_{reg} be the vertex operator algebra extension of $L(1, 0)$ corresponding to the commutative algebra object $\widehat{\Phi}(\mathcal{O}(G))$. Then there is a G -equivariant isomorphism of VOA extensions*

$$U_{\text{reg}} \xrightarrow{\sim} V_{A_1},$$

where the action on U_{reg} is induced by right translations on $\mathcal{O}(G)$ and the action on V_{A_1} is the standard automorphism action.

Proof. As a $G_{\text{left}} \times G_{\text{right}}$ -module, Peter–Weyl gives

$$\mathcal{O}(G) \cong \bigoplus_{m \geq 0} W_{2m}^* \otimes W_{2m}.$$

Applying Φ to the left factor gives

$$\widehat{\Phi}(\mathcal{O}(G)) \cong \bigoplus_{m \geq 0} X_m \otimes W_{2m}$$

as an $L(1, 0) \times G_{\text{right}}$ -module. After forgetting the right action, its underlying Virasoro object is

$$\bigoplus_{m \geq 0} (2m + 1)L(1, m^2).$$

By Lemma 2.16, it defines the grading-restricted VOA extension U_{reg} . The algebra $\mathcal{O}(G)$ has no nonzero proper left- G -stable ideals because G acts transitively on itself; Proposition 2.13 shows that U_{reg} is simple.

McRae–Yang write $L_{r,s}^{(1)}$ for the relevant central-charge-one Virasoro modules. Their lowest-weight formula gives

$$h_{2m+1,1}^{(1)} = m^2, \quad L_{2m+1,1}^{(1)} = L(1, m^2) = X_m.$$

Thus the preceding decomposition is exactly the one in [20, Theorem B.1]. That theorem identifies the simple extension with

$$L_1(\mathfrak{sl}_2) \cong V_{A_1}.$$

Apply Lemma 2.11 to the $G_{\text{left}} \times G_{\text{right}}$ -algebra $\mathcal{O}(G)$. The right action therefore becomes a rational algebraic G -action by commutative-algebra-object automorphisms and, under the extension correspondence, by VOA automorphisms. By the displayed bicovariant decomposition, this action has the $G \times L(1, 0)$ -module structure required in [20, Theorem B.2]. That theorem allows the isomorphism supplied by [20, Theorem B.1] to be chosen G -equivariantly. We fix one such isomorphism for the remainder of the paper. $\square$

Corollary 2.18 (Fixed-point/homogeneous-space dictionary). *Let $H \leq G$ be reductive and suppose G/H is affine. Then the VOA extension associated with $\widehat{\Phi}(\mathcal{O}(G/H))$ is isomorphic to*

$$V_{A_1}^H.$$

Conversely, any extension whose algebra object corresponds to $\mathcal{O}(G/H)$ is isomorphic to $V_{A_1}^H$.

Proof. By Theorem 2.17, right translations by H correspond to the automorphism action on V_{A_1} . Hence

$$\mathcal{O}(G/H) = \mathcal{O}(G)^{H_{\text{right}}}.$$

Algebraic Peter–Weyl gives, with the left and right actions on the two factors,

$$\mathcal{O}(G) = \bigoplus_{m \geq 0} W_{2m}^* \otimes W_{2m}.$$

Consequently

$$\mathcal{O}(G)^{H_{\text{right}}} = \bigoplus_{m \geq 0} W_{2m}^* \otimes W_{2m}^H.$$

Lemma 2.11, applied to the commuting right H -action, gives directly

$$\widehat{\Phi}(\mathcal{O}(G)^{H_{\text{right}}}) \cong \widehat{\Phi}(\mathcal{O}(G))^H$$

as commutative algebra objects. Under the equivariant isomorphism of Theorem 2.17, the right-hand side is the fixed-point VOA $V_{A_1}^H$. The converse follows from the extension/algebra correspondence and Lemma 2.16. $\square$

3. SQUARE CORES AND THE EXCEPTIONAL CHARACTER BRANCHES

3.1. The invariant form and the first primary. Throughout this section V is a simple self-contragredient VOA of CFT type and central charge one. Li identifies the space of invariant bilinear forms with $(V_0/L(1)V_1)^*$ [14, Corollary 3.2]. For a simple CFT-type VOA, self-contragredience is equivalent to $L(1)V_1 = 0$; the normalized nonzero invariant form is then symmetric and nondegenerate. We fix the normalization $(\mathbf{1}, \mathbf{1}) = 1$. Since $L(0)$ is self-adjoint, distinct conformal-weight spaces are orthogonal, and the form restricts nondegenerately to every V_n .

Definition 3.1. *For $h \in \mathbb{Z}_{\geq 0}$ set*

$$P_h(V) = \{v \in V_h \mid L(n)v = 0 \text{ for all } n > 0\}.$$

Lemma 3.2 (Primary complement at the first deviation). *Let $h > 0$ and suppose that*

$$V_n = L(1, 0)_n \quad (0 \leq n < h).$$

Then

$$V_h = L(1, 0)_h \perp P_h(V).$$

In particular, the restriction of the invariant form to $P_h(V)$ is nondegenerate and

$$\dim P_h(V) = \dim V_h - \dim L(1, 0)_h.$$

Proof. The invariant form restricts nondegenerately to the simple conformal subVOA $L(1, 0)$, and hence to each of its homogeneous subspaces. Let $Q = L(1, 0)_h^\perp \cap V_h$. If $v \in Q$, then, for $1 \leq n \leq h$ and $w \in V_{h-n} = L(1, 0)_{h-n}$, invariance gives

$$(L(n)v, w) = (v, L(-n)w) = 0.$$

The form on V_{h-n} is nondegenerate, so $L(n)v = 0$. For $n > h$ the target has negative conformal weight and vanishes. Thus $Q \subseteq P_h(V)$.

Conversely, let $v \in P_h(V)$. Every vector in $L(1, 0)_h$ is a linear combination of vacuum descendants

$$L(-n_1) \cdots L(-n_r) \mathbf{1}, \quad n_i > 0.$$

Repeated invariance moves the negative modes to positive modes acting on v , and therefore pairs each such descendant trivially with v . Hence $P_h(V) \subseteq Q$. The asserted orthogonal decomposition and the dimension formula follow. $\square$

Lemma 3.3 (First-primary extraction). *Let $h > 0$. Assume*

$$V_n = L(1, 0)_n \quad (0 \leq n < h), \quad \dim V_h = \dim L(1, 0)_h + 1.$$

Then there is, uniquely up to nonzero scalar, a vector $v \in P_h(V)$ such that

$$V_h = L(1, 0)_h \oplus \mathbb{C}v, \quad (v, v) \neq 0.$$

Proof. Lemma 3.2 shows that $P_h(V)$ is a one-dimensional nondegenerate orthogonal complement to $L(1, 0)_h$. Choose $0 \neq v \in P_h(V)$. Then $(v, v) \neq 0$ and the required direct sum holds.

For uniqueness, let v' be another primary vector giving such a complement. Then

$$v' = av + w$$

for some $a \in \mathbb{C}^\times$ and $w \in L(1, 0)_h$. The vector $w = v' - av$ is primary. A positive-weight primary in the simple highest-weight module $L(1, 0)$ would generate a nonzero proper submodule, so $w = 0$. Hence v' is a nonzero scalar multiple of v . $\square$

Lemma 3.4 (The vacuum channel). *Let $u, v \in P_h(V)$, where $h \in \mathbb{Z}_{\geq 0}$. Then*

$$u_{2h-1}v = (-1)^h(u, v)\mathbf{1}.$$

In particular, the projection of the product of two primary vectors to the vacuum module is nonzero whenever their invariant pairing is nonzero.

Proof. The invariance identity for the bilinear form gives

$$(Y(u, z)v, \mathbf{1}) = (v, Y(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})\mathbf{1}).$$

Since u is primary of integral weight h , the right-hand side is

$$(-1)^h z^{-2h}(v, e^{z^{-1}L(-1)}u) = (-1)^h z^{-2h}(v, u).$$

Indeed, every term involving a positive power of $L(-1)$ has weight greater than h and is orthogonal to v . Comparing the coefficient of z^{-2h} proves the formula. $\square$

3.2. The first singular vector. To distinguish Virasoro Verma modules from the Heisenberg modules used later, write $M_{\text{Vir}}(1, h)$ for the Verma module of central charge one and highest weight h .

Proposition 3.5 (The first singular vector at central charge one). *For every $m \geq 0$, the Verma module $M_{\text{Vir}}(1, m^2)$ has, up to scalar, a unique singular vector at level $2m + 1$. This is its first nontrivial singular vector, and it generates the unique maximal proper submodule, which is isomorphic to $M_{\text{Vir}}(1, (m + 1)^2)$. Equivalently, there is an exact sequence*

$$0 \longrightarrow M_{\text{Vir}}(1, (m + 1)^2) \longrightarrow M_{\text{Vir}}(1, m^2) \longrightarrow L(1, m^2) \longrightarrow 0.$$

The embedded Verma module is also the radical of the normalized Shapovalov form on $M_{\text{Vir}}(1, m^2)$.

Proof. This is [16, Proposition 2.1], with the parameter there specialized to the even integer $2m$; see also the Feigin–Fuchs embedding description in [15]. The singular level is $2m + 1$, so its absolute conformal weight is

$$m^2 + (2m + 1) = (m + 1)^2.$$

For a Verma module, the radical of the normalized Shapovalov form is its unique maximal proper submodule, which proves the last assertion. $\square$

Lemma 3.6 (First-singular saturation). *Fix $m \geq 1$ and set*

$$h = m^2, \quad H = (m + 1)^2.$$

Assume

- (i) $V_n = L(1, 0)_n$ for $0 \leq n < h$;
- (ii) $\dim V_h = \dim L(1, 0)_h + 1$;
- (iii) for some $\varepsilon \in \{0, 1\}$,

$$\dim V_H = \dim L(1, 0)_H + \dim(X_m)_H + \varepsilon.$$

Then the vector v from Lemma 3.3 generates

$$U(\text{Vir})v \cong X_m.$$

Proof. Let $C = U(\text{Vir})v$ and let

$$\phi : M_{\text{Vir}}(1, m^2) \twoheadrightarrow C$$

be the canonical Virasoro homomorphism sending the Verma highest-weight vector to v . Let $\tilde{s} \in M_{\text{Vir}}(1, m^2)_H$ be a nonzero first singular vector and put $s = \phi(\tilde{s})$. If $s = 0$, Proposition 3.5 shows that ϕ factors through X_m . The induced map $X_m \rightarrow C$ is nonzero and surjective; since X_m is simple, it is an isomorphism. Suppose therefore that $s \neq 0$.

By Proposition 3.5, there is no singular vector in $M_{\text{Vir}}(1, m^2)$ below weight H , and the singular space at weight H is one-dimensional. If the kernel of the map to C first became nonzero below H , a vector of least level in that kernel would be singular, which is impossible. At weight H , the same argument shows that the kernel is contained in the first-singular line. Since s survives, the map is injective through weight H . Consequently

$$\dim C_H = \dim M_{\text{Vir}}(1, m^2)_H = \dim(X_m)_H + 1.$$

Let $\langle \cdot, \cdot \rangle_{\text{Sh}}$ be the Shapovalov form on $M_{\text{Vir}}(1, m^2)$, normalized to be nonzero on the highest-weight line. Its radical is the unique maximal proper submodule and, by Proposition 3.5, contains $\tilde{s}$. Pull the invariant form on V back along ϕ by setting

$$\langle x, y \rangle_\phi = (\phi(x), \phi(y)).$$

Since $(v, v) \neq 0$, this is a nonzero contravariant form on the Verma module and hence a nonzero scalar multiple of the Shapovalov form. Therefore, for every $x \in M_{\text{Vir}}(1, m^2)$,

$$(s, \phi(x)) = (\phi(\tilde{s}), \phi(x)) = \langle \tilde{s}, x \rangle_\phi = 0.$$

Surjectivity of ϕ gives $(s, C) = 0$. The quotient map is Virasoro-linear. Thus, if $s \neq 0$, it is still annihilated by every $L(n)$ with $n > 0$; in particular, s is primary. Moreover $C \cap L(1, 0) = 0$. Indeed, the intersection is a Virasoro submodule of the simple module $L(1, 0)$, so it is either zero or all of $L(1, 0)$. The second possibility is impossible because C is supported in conformal weights at least $m^2 > 0$, whereas $L(1, 0)$ contains the vacuum in weight zero.

If $\varepsilon = 0$, the displayed dimension formula gives

$$\dim V_H \geq \dim L(1, 0)_H + \dim(X_m)_H + 1,$$

contradicting (iii).

Assume $\varepsilon = 1$. Together with $C \cap L(1, 0) = 0$, the displayed dimension formula and condition (iii) force

$$V_H = L(1, 0)_H \oplus C_H.$$

We already know $s \perp C_H$. Since s is primary, it is also orthogonal to every vacuum descendant: if

$$w = L(-n_1) \cdots L(-n_r) \mathbf{1} \in L(1, 0)_H,$$

then repeated invariance moves the negative Virasoro modes to positive modes acting on s , giving $(s, w) = 0$. Thus $s \perp V_H$, contradicting nondegeneracy of the invariant form on V_H . Hence $s = 0$. $\square$

3.3. Semisimple square cores. The following closure statement replaces any appeal to global Virasoro semisimplicity.

Theorem 3.7 (Semisimple square-core closure). *Let V be a grading-restricted VOA extension of $L(1, 0)$. Let $S \subset V$ be a finite direct sum of embedded square simple Virasoro modules X_m , containing $X_0 = L(1, 0)$, and let*

$$U = \langle S \rangle_{\text{VOA}}.$$

Then

$$U \in \text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}}).$$

Equivalently, U is an algebraic direct sum of square simples as a Virasoro module. No nonsplit square-block extension is created by the vertex-algebra multiplication.

Proof. Set $F_0 = S$. Suppose F_r has been constructed and choose an internal decomposition

$$F_r = \bigoplus_{\lambda \in \Lambda_r} S_\lambda, \quad S_\lambda \cong X_{m(\lambda)}.$$

Here λ records the actual embedded copy, not only its isomorphism class. For every ordered pair $\lambda, \mu \in \Lambda_r$, let

$$I_{\lambda, \mu} = \text{CoeffIm} \left(Y_V|_{S_\lambda \otimes S_\mu} \right).$$

Lemma 2.7 and Theorem 2.3 show that $I_{\lambda, \mu}$ is a finite semisimple square object. Define

$$F_{r+1} = F_r + \sum_{\lambda, \mu \in \Lambda_r} I_{\lambda, \mu} \subset V.$$

Consider the natural Virasoro-module epimorphism

$$\psi_r : F_r \oplus \bigoplus_{\lambda, \mu \in \Lambda_r} I_{\lambda, \mu} \twoheadrightarrow F_{r+1}, \quad \psi_r(x, (y_{\lambda, \mu})) = x + \sum_{\lambda, \mu} y_{\lambda, \mu}.$$

Its source is a finite semisimple object of $\mathcal{C}_{\text{sq}}$. By Lemma 2.8, the ambient Virasoro submodule $\ker \psi_r$ is again a finite direct sum of square simples. Hence the quotient F_{r+1} belongs to $\mathcal{C}_{\text{sq}}$. We have therefore constructed an increasing sequence

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots.$$

Every bracketed mode word of binary-tree depth at most r in vectors of S lies in F_r . Indeed, write two vectors of F_r in terms of the actual summands S_λ . Bilinearity and the definition of $I_{\lambda, \mu}$ put every coefficient of their vertex product in F_{r+1} ; Virasoro descendants remain there because F_{r+1} is a Virasoro submodule. This proves the claim by induction. Conversely, each coefficient image used in the recursion is contained in the vertex subalgebra generated by S . Therefore

$$U = \bigcup_{r \geq 0} F_r,$$

a directed union of objects of $\mathcal{C}_{\text{sq}}$, and $U \in \text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}})$.

There is also an explicit splitting. Since F_{r+1} is semisimple, its Virasoro submodule F_r is a direct summand; choose C_r with

$$F_{r+1} = F_r \oplus C_r.$$

Induction gives

$$U = F_0 \oplus \bigoplus_{r \geq 0} C_r.$$

Every C_r is a finite direct sum of square simples. Hence U is an algebraic direct sum of the X_j , and no nonsplit square block occurs. $\square$

3.4. A closed-orbit reconstruction lemma. We transport square cores through the equivalence $\widehat{\Phi}^{-1} : \text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}}) \rightarrow \text{Rep}^{\text{rat}} G$.

Lemma 3.8 (Generation under a symmetric equivalence). *Let V_0 be a vertex operator algebra, let G be a reductive algebraic group, and let*

$$\Phi : \text{Rep } G \xrightarrow{\sim} \mathcal{C}$$

be a symmetric tensor equivalence for which the extension/algebra correspondence is available in $\text{Ind}_{\text{CMY}}(\mathcal{C})$. Assume that it extends to an exact symmetric monoidal equivalence

$$\widehat{\Phi} : \text{Rep}^{\text{rat}} G \xrightarrow{\sim} \text{Ind}_{\text{CMY}}(\mathcal{C}).$$

Suppose that a VOA extension $U \in \text{Ind}_{\text{CMY}}(\mathcal{C})$ is generated over V_0 by a finite subobject $E \in \mathcal{C}$. Then the corresponding commutative rational G -algebra $B = \widehat{\Phi}^{-1}(U)$ is generated, as an ordinary commutative $\mathbb{C}$ -algebra, by the finite-dimensional G -module $\Phi^{-1}(E)$. In particular, B is finitely generated.

Proof. Put $W = \Phi^{-1}(E)$, and let $B_W \subseteq B$ be the commutative algebra subobject generated by the unit and W . Its image

$$C := \widehat{\Phi}(B_W) \subseteq U$$

is an $L(0)$ -stable submodule of the grading-restricted VOA U and is therefore grading restricted. Proposition 2.14 shows that C is a vertex operator subalgebra of U containing V_0 and E . The generating hypothesis forces the inclusion $C \hookrightarrow U$ to be an isomorphism. This inclusion is $\widehat{\Phi}$ applied to $B_W \hookrightarrow B$, and an equivalence reflects isomorphisms. Hence $B_W \hookrightarrow B$ is an isomorphism. Finally, commutative algebra objects of $\text{Rep}^{\text{rat}} G$ are ordinary rational G -algebras, and B_W is the image of the symmetric algebra $\text{Sym}(W)$. Hence W generates B as an ordinary commutative algebra, proving the claim. $\square$

Lemma 3.9 (First invariant degrees). *Let W_{2m} be the spin- m irreducible representation of $G = \text{SO}_3(\mathbb{C})$. The first positive m for which $W_{2m}^H \neq 0$ is*

H	T	$N(T)$	C_n	D_n	A_4	S_4	A_5
first m	1	2	1	2	3	4	6.

Here C_n denotes a finite cyclic rotation subgroup, and D_n a dihedral rotation subgroup with $n \geq 2$.

Proof. For T , the zero-weight line of W_{2m} is nonzero for every $m \geq 1$. The nontrivial Weyl element acts on this line by $(-1)^m$, so the first $N(T)$ -invariant occurs at $m = 2$. A finite cyclic rotation group fixes its rotation axis in the spin-one representation, so its first invariant degree is 1. A dihedral rotation group D_n with $n \geq 2$ has no common fixed vector in spin one: a half-turn about an axis perpendicular to the principal rotation axis changes the sign of that axis. On the other hand, the traceless quadratic tensor determined by the principal axis is fixed by the whole dihedral group, so its first invariant degree is 2.

For the three Platonic groups, use the rotation character

$$\chi_m(\theta) = 1 + 2 \sum_{j=1}^m \cos(j\theta) = \frac{\sin((m + \frac{1}{2})\theta)}{\sin(\theta/2)}$$

and the averaging formula

$$\dim W_{2m}^H = \frac{1}{|H|} \sum_{h \in H} \chi_m(h).$$

The angle distributions are

$$\begin{array}{l|l} A_4 & 1 \cdot 0, \ 3 \cdot \pi, \ 8 \cdot \frac{2\pi}{3}, \\ S_4 & 1 \cdot 0, \ 9 \cdot \pi, \ 8 \cdot \frac{2\pi}{3}, \ 6 \cdot \frac{\pi}{2}, \\ A_5 & 1 \cdot 0, \ 15 \cdot \pi, \ 20 \cdot \frac{2\pi}{3}, \ 12 \cdot \frac{2\pi}{5}, \ 12 \cdot \frac{4\pi}{5}. \end{array}$$

Direct substitution gives the first positive invariant degrees 3, 4, 6. □

Lemma 3.10 (Positive-dimensional reductive subgroups of $\mathrm{PSL}_2(\mathbb{C})$). *Let H be a proper closed reductive subgroup of $G = \mathrm{PSL}_2(\mathbb{C})$. If $\dim H > 0$, then, up to conjugacy,*

$$H = T \quad \text{or} \quad H = N(T)$$

for a maximal torus T of G .

Proof. The Lie algebra $\mathfrak{h}$ of H° is a proper reductive subalgebra of $\mathfrak{sl}_2$. Its semisimple part must be zero, since a nonzero semisimple subalgebra of $\mathfrak{sl}_2$ is all of $\mathfrak{sl}_2$. Thus $\mathfrak{h}$ is a nonzero toral subalgebra, necessarily a Cartan subalgebra. It follows that $H^\circ = T$ is a maximal torus. Because H° is characteristic in H , one has $H \subseteq N(T)$. Finally,

$$H/T \leq N(T)/T \cong \mathbb{Z}_2,$$

which gives the two possibilities. □

Theorem 3.11 (One-generator closed-orbit reconstruction). *Let $m \in \{3, 4, 6\}$. Suppose V contains a nondegenerate primary v generating an embedded X_m , and assume*

$$P_{j^2}(V) = 0 \quad (0 < j < m).$$

Let

$$U = \langle L(1, 0), X_m \rangle_{\text{VOA}}.$$

Then $U \in \text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}})$ and its commutative G -algebra B is finitely generated. Moreover $\text{Spec } B$ has a unique closed G -orbit G/H , and restriction to that orbit gives a G -algebra epimorphism

$$B \twoheadrightarrow \mathcal{O}(G/H),$$

and

$$H \text{ is conjugate in } G \text{ to } \begin{cases} A_4, & m = 3, \\ S_4, & m = 4, \\ A_5, & m = 6. \end{cases}$$

Proof. Theorem 3.7 gives $U \in \text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}})$ and Lemma 3.8 gives finite generation of B by $E = W_{2m}$. Under the tensor equivalence,

$$\dim B^G = \dim \text{Hom}_G(\mathbf{1}, B) = \dim \text{Hom}_{\text{Vir}}(X_0, U).$$

A Virasoro homomorphism $X_0 = L(1, 0) \rightarrow U$ is determined by the image of the vacuum top vector, and that image must lie in $U_0 = \mathbb{C}\mathbf{1}$. Therefore

$$B^G = \mathbb{C}\mathbf{1}.$$

Let $N = \sqrt{0} \subset B$. This is a G -stable ideal. We first show that

$$(B/N)^G = \mathbb{C}.$$

Let $\bar{b} \in (B/N)^G$ and choose a lift $b \in B$. Because B is a rational G -module, b lies in a finite-dimensional rational G -submodule $M \subset B$. If $\bar{M}$ is the image of M in B/N , then

$$0 \longrightarrow M \cap N \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0$$

is an exact sequence of finite-dimensional rational G -modules. Reductivity over $\mathbb{C}$ makes this sequence split, so the invariant vector $\bar{b}$ has a G -invariant lift in M . Thus $B^G \rightarrow (B/N)^G$ is surjective. Moreover $N^G = 0$, since $N^G \subset B^G = \mathbb{C}$ and no nonzero constant is nilpotent. The claim follows.

Every prime ideal of B contains N . Hence the closed immersion

$$\mathrm{Spec}(B/N) \longrightarrow \mathrm{Spec} B$$

is a homeomorphism on underlying topological spaces. Since N is G -stable, this homeomorphism is G -equivariant. It identifies the G -orbits and their closures, so the two spectra have the same closed G -orbits.

Set $X = \mathrm{Spec}(B/N)$. It is a nonempty reduced affine G -variety of finite type, and

$$X//G = \mathrm{Spec}((B/N)^G) = \mathrm{Spec} \mathbb{C}.$$

The affine GIT closed-orbit theorem says that each fiber of the quotient map contains a unique closed G -orbit [17, Chapter 1, §1]. Hence X has a unique closed orbit $\mathcal{O} \cong G/H$. Since $\mathcal{O}$ is closed in the affine variety X , it is affine; Matsushima's criterion implies that H is reductive [18]. The closed immersion $\mathcal{O} \hookrightarrow X$ induces a surjection $B/N \twoheadrightarrow \mathcal{O}(G/H)$. Composing with $B \twoheadrightarrow B/N$ gives the required G -algebra map

$$B \twoheadrightarrow \mathcal{O}(G/H).$$

We next prove that the generating copy $E = W_{2m}$ survives on the closed orbit. Let $p_0 : B \rightarrow B^G = \mathbb{C}\mathbf{1}$ be the projection onto the trivial isotypic component, and set

$$\beta = p_0 \circ \mu_B|_{E \otimes E} : E \otimes E \longrightarrow B^G.$$

Let $\pi_0 : U \rightarrow X_0$ be the morphism corresponding under $\widehat{\Phi}$ to p_0 , and let I_{μ_U} be the $P(1)$ -intertwining map associated with the multiplication of U . Compatibility of the symmetric tensor equivalence with the canonical $P(1)$ -maps identifies β with

$$\pi_0 \circ \mu_U|_{X_m \boxtimes X_m}.$$

For the chosen highest-weight vector $v \in X_m$, of weight $h = m^2$, the weight-zero projection is

$$\mathrm{pr}_{U_0} I_{\mu_U}(v \otimes v) = v_{2h-1}v = (-1)^h(v, v)\mathbf{1} \neq 0$$

by Lemma 3.4. It follows that $\beta \neq 0$. Commutativity makes β symmetric, and, since E is irreducible and self-dual, this nonzero invariant bilinear form is nondegenerate. Let

$$q_\beta \in (\mathrm{Sym}^2 E)^G$$

be its inverse Casimir tensor, normalized by

$$\beta(q_\beta) = (\dim E)\mathbf{1} = (2m+1)\mathbf{1}.$$

The tensor q_β is invariant and μ_B is G -equivariant, so $\mu_B(q_\beta) \in B^G$. The projection p_0 is therefore the identity on this element, and

$$\mu_B(q_\beta) = p_0(\mu_B(q_\beta)) = \beta(q_\beta) = (2m+1)\mathbf{1} \neq 0.$$

Let $\rho : B \rightarrow \mathcal{O}(G/H)$ be the restriction map. If $\rho|_E = 0$, then $(\rho \otimes \rho)(q_\beta) = 0$, and the algebra homomorphism property gives

$$\rho(\mu_B(q_\beta)) = \mu_{\mathcal{O}(G/H)}((\rho \otimes \rho)(q_\beta)) = 0.$$

On the other hand, the preceding identity gives

$$\rho(\mu_B(q_\beta)) = (2m+1)\mathbf{1} \neq 0,$$

a contradiction. Thus the restriction map is nonzero on E . Algebraic Frobenius reciprocity gives

$$\mathrm{Hom}_G(E, \mathcal{O}(G/H)) \cong (E^*)^H.$$

Since E is self-dual, it follows that $E^H \neq 0$.

Suppose $W_{2j}^H \neq 0$ for some $0 < j < m$. Peter–Weyl then shows that $\mathcal{O}(G/H)$ contains a copy of W_{2j} . Choose finitely many preimages in B of a basis of that copy. Because B is a rational G -module, the G -submodule generated by those finitely many vectors is finite dimensional. Its map onto W_{2j} splits because finite-dimensional rational representations of reductive G are semisimple. Hence B contains a copy of W_{2j} , and therefore U contains an embedded X_j . Its highest-weight vector is a primary of weight j^2 , contrary to the hypothesis. Thus m is the first positive invariant degree of H .

The subgroup H is proper because $E^H \neq 0$ and E is nontrivial. If H has positive dimension, Lemma 3.10 gives $H = T$ or $N(T)$. Their first invariant degrees are 1 and 2, and hence neither can occur here. If H is finite, the classical classification of finite subgroups of $\mathrm{PSL}_2(\mathbb{C})$, up to conjugacy, gives the cyclic, dihedral, tetrahedral, octahedral, and icosahedral groups. Lemma 3.9 now identifies H , up to conjugacy, as A_4 , S_4 , or A_5 according as $m = 3, 4$, or 6 . $\square$

3.5. Polyhedral multiplicities and exceptional rigidity. For later use we record the first spherical multiplicities. For $H = A_4, S_4, A_5$ set

$$a_m(H) = \dim W_{2m}^H.$$

The character average in the proof of Lemma 3.9 gives

$H \setminus m$	1	2	3	4	5	6	7
A_4	0	0	1	1	0	2	1
S_4	0	0	0	1	0	1	0
A_5	0	0	0	0	0	1	0

Summing the character averages uses

$$\sum_{m \geq 0} \chi_m(\theta) t^m = \frac{1+t}{1-2t \cos \theta + t^2}.$$

Substitution of the three angle distributions and elementary simplification give the Hilbert series

$$\begin{aligned} \sum_{m \geq 0} a_m(A_4) t^m &= \frac{1-t^{12}}{(1-t^3)(1-t^4)(1-t^6)}, \\ \sum_{m \geq 0} a_m(S_4) t^m &= \frac{1-t^{18}}{(1-t^4)(1-t^6)(1-t^9)}, \\ \sum_{m \geq 0} a_m(A_5) t^m &= \frac{1-t^{30}}{(1-t^6)(1-t^{10})(1-t^{15})}. \end{aligned}$$

Corollary 3.12. *Assume*

$$\text{ch } V = \text{ch } V_{A_1}^H, \quad H \in \{A_4, S_4, A_5\}.$$

Then the first positive primary occurs respectively at weights

$$9, \quad 16, \quad 36,$$

and generates respectively

$$X_3, \quad X_4, \quad X_6.$$

Proof. Write

$$V_{A_1}^H \cong \bigoplus_{j \geq 0} a_j(H) X_j$$

and let $m = 3, 4, 6$ for $H = A_4, S_4, A_5$, respectively. The table gives

$$a_j(H) = 0 \quad (0 < j < m), \quad a_m(H) = 1.$$

Since the lowest conformal weight of X_j is j^2 , the character identity and the inclusion $L(1, 0) \subset V$ imply

$$V_n = L(1, 0)_n \quad (0 \leq n < m^2), \quad \dim V_{m^2} = \dim L(1, 0)_{m^2} + 1.$$

Lemma 3.3 therefore supplies a nondegenerate primary of weight m^2 .

Set $H_m = (m+1)^2$. At conformal weight H_m , no summand X_j with $j > m+1$ contributes, while the positive summands with $j \leq m+1$ are the single copy of X_m and $a_{m+1}(H)$ copies of X_{m+1} . The top level of X_{m+1} is one-dimensional, and hence

$$\dim V_{H_m} = \dim L(1, 0)_{H_m} + \dim(X_m)_{H_m} + a_{m+1}(H).$$

Now

$$a_4(A_4) = 1, \quad a_5(S_4) = 0, \quad a_7(A_5) = 0.$$

Lemma 3.6 applies with $\varepsilon = 1, 0, 0$ in the three cases, proving that the cyclic modules are X_3, X_4, X_6 . $\square$

Theorem 3.13 (Exceptional character rigidity). *Let $H \in \{A_4, S_4, A_5\}$. Let V be a simple self-contragredient VOA of CFT type and central charge one. If*

$$\text{ch } V = \text{ch } V_{A_1}^H,$$

then

$$V \cong V_{A_1}^H.$$

No Virasoro complete reducibility hypothesis on V is required.

Proof. Let $m_0 = 3, 4, 6$ for $H = A_4, S_4, A_5$, respectively. By Corollary 3.12, V contains a nondegenerate embedded X_{m_0} and no positive square primary of smaller degree. Set

$$U = \langle L(1, 0), X_{m_0} \rangle_{\text{VOA}}.$$

Theorem 3.11 produces a commutative G -algebra B and a surjection

$$B \twoheadrightarrow \mathcal{O}(G/H'),$$

where H' is conjugate in G to the prescribed subgroup H . Conjugation identifies the two homogeneous spaces equivariantly, so after this identification we have a surjection $B \twoheadrightarrow \mathcal{O}(G/H)$. Write

$$U \cong \bigoplus_{j \geq 0} b_j X_j, \quad V_{A_1}^H \cong \bigoplus_{j \geq 0} a_j(H) X_j.$$

The multiplicities b_j are finite: each copy of X_j contributes an independent lowest-weight vector to the finite-dimensional space $U_{j^2} \subseteq V_{j^2}$. Thus the character comparison below is legitimate coefficient by coefficient. For each j , Peter–Weyl gives $a_j(H)$ copies of W_{2j} in $\mathcal{O}(G/H)$. The surjection $B \twoheadrightarrow \mathcal{O}(G/H)$ implies $b_j \geq a_j(H)$. To see this without using semisimplicity of arbitrary infinite-dimensional rational modules, take the finite-dimensional

target isotypic component and finitely many preimages; the finite-dimensional G -submodule they generate maps onto it and splits.

All coefficients in the characters of X_j are nonnegative. Hence

$$\text{ch } U \geq \text{ch } V_{A_1}^H$$

coefficientwise. On the other hand $U \subset V$ and $\text{ch } V = \text{ch } V_{A_1}^H$, so

$$\text{ch } U \leq \text{ch } V = \text{ch } V_{A_1}^H.$$

Therefore equality holds throughout. In particular $U_n = V_n$ for every n , so $U = V$. Put

$$K = \ker(B \twoheadrightarrow \mathcal{O}(G/H)).$$

Exactness of $\widehat{\Phi}$ gives

$$0 \longrightarrow \widehat{\Phi}(K) \longrightarrow U \longrightarrow \widehat{\Phi}(\mathcal{O}(G/H)) \longrightarrow 0.$$

By Corollary 2.18, the last term is $V_{A_1}^H$ as an $L(1, 0)$ -module. Taking the finite-dimensional conformal-weight- n spaces therefore gives

$$\dim \widehat{\Phi}(K)_n = \dim U_n - \dim (V_{A_1}^H)_n = 0 \quad (n \geq 0).$$

Thus $\widehat{\Phi}(K) = 0$. Since $\widehat{\Phi}$ is an equivalence, $K = 0$, and hence

$$B \cong \mathcal{O}(G/H).$$

Corollary 2.18 now gives $V \cong V_{A_1}^H$. □

Corollary 3.14 (The icosahedral gap). *Under the hypotheses of Theorem 1.1, if*

$$P_h(V) = 0 \quad (0 < h < 36),$$

then $V \cong V_{A_1}^{A_5}$.

Proof. By Theorem 2.2, the vacuum character is one of the five classical types. A lattice VOA already deviates from the vacuum Virasoro character at weight one, since its Heisenberg vector is nonzero while $L(1, 0)_1 = 0$.

For the orbifold branch, write $L = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 2k$. If $k \leq 4$, the vector $e^\alpha + e^{-\alpha} \in V_L^+$ contributes outside the charge-zero vacuum module in weight k . If $k > 4$, choose a normalized Heisenberg generator h . The charge-zero subalgebra $M(1)^+$ contains the nonzero vector

$$J = h(-1)^4 \mathbf{1} - 2h(-3)h(-1) \mathbf{1} + \frac{3}{2}h(-2)^2 \mathbf{1}.$$

With the normalization $[h(r), h(s)] = r\delta_{r+s,0}$, the commutator $[L(n), h(-r)] = rh(n-r)$ gives

$$\begin{aligned} L(1)(h(-3)h(-1)\mathbf{1}) &= 3h(-2)h(-1)\mathbf{1}, & L(1)h(-2)^2\mathbf{1} &= 4h(-2)h(-1)\mathbf{1}, \\ L(2)h(-1)^4\mathbf{1} &= 6h(-1)^2\mathbf{1}, & L(2)(h(-3)h(-1)\mathbf{1}) &= 3h(-1)^2\mathbf{1}, \\ L(4)(h(-3)h(-1)\mathbf{1}) &= 3\mathbf{1}, & L(4)h(-2)^2\mathbf{1} &= 4\mathbf{1}. \end{aligned}$$

The omitted terms in these three rows vanish, so the coefficients in J give $L(1)J = L(2)J = L(4)J = 0$. The same commutator formula gives $L(3)J = 0$, and $L(n)J = 0$ for $n > 4$ by weight. Thus J is a positive-weight primary and cannot lie in the simple vacuum Virasoro module. Hence every rank-one V_L^+ character deviates by weight four.

The A_4 and S_4 characters first deviate at weights 9 and 16. At any first deviation weight h , all lower homogeneous spaces agree with the vacuum module and

$$\dim V_h - \dim L(1, 0)_h > 0.$$

Lemma 3.2 therefore gives $P_h(V) \neq 0$, irrespective of the size of the dimension jump. The gap hypothesis excludes all of these branches, leaving the A_5 character. Theorem 3.13 finishes the proof. $\square$

4. THE WEIGHT-FOUR CORE AND RECONSTRUCTION OF $M(1)^+$

4.1. The spin-two matrix model. Let

$$E = W_4 \cong \text{Sym}_0^2(\mathbb{C}^3)$$

be the five-dimensional spin-two representation of $G \cong \text{SO}_3(\mathbb{C})$, realized as traceless symmetric 3×3 matrices with the action by orthogonal conjugation. Let

$$\iota_E : E^* \xrightarrow{\sim} E$$

be the self-duality defined by the invariant trace form. Define the quadratic polynomial and quadratic covariant

$$q(A) = \text{tr}(A^2), \quad Q(A) = A^2 - \frac{1}{3} \text{tr}(A^2)I_3.$$

Then q spans $(\text{Sym}^2 E^*)^G$, and, after polarization, Q spans $\text{Hom}_G(\text{Sym}^2 E, E)$; equivalently,

$$\text{Sym}^2 E \cong W_0 \oplus W_4 \oplus W_8.$$

Lemma 4.1 (The quadratic-covariant orbit). *Let $c \in \mathbb{C}^\times$ and $\lambda \in \mathbb{C}$. Consider the affine scheme*

$$Z_{c,\lambda} = \{A \in \text{Sym}_0^2(\mathbb{C}^3) : q(A) = c, Q(A) = \lambda A\}.$$

If $Z_{c,\lambda}$ is nonempty, then

$$\lambda \neq 0, \quad c = 6\lambda^2,$$

and

$$Z_{c,\lambda} = G \cdot \text{diag}(2\lambda, -\lambda, -\lambda) \cong G/N(T)$$

scheme-theoretically. In particular $Z_{c,\lambda}$ is smooth and reduced.

Proof. The equation $Q(A) = \lambda A$ is equivalent to

$$(4.1) \quad A^2 - \lambda A - \frac{c}{3}I_3 = 0.$$

If $\lambda = 0$, then $A^2 = (c/3)I_3$. Since $c \neq 0$, all eigenvalues are $\pm\sqrt{c/3}$, and three such eigenvalues cannot sum to zero. Thus $\lambda \neq 0$.

Equation (4.1) shows that A has at most two eigenvalues. Since A is traceless and nonzero, the multiplicities are 1 and 2. Let the eigenvalues be a and b , with $a + 2b = 0$. Comparing with the roots of the quadratic polynomial in (4.1) gives

$$a + b = \lambda, \quad ab = -\frac{c}{3}.$$

Hence

$$b = -\lambda, \quad a = 2\lambda, \quad c = 6\lambda^2.$$

The roots are distinct, so A is semisimple, and its two eigenspaces are mutually orthogonal. The one-dimensional 2λ -eigenspace is nondegenerate: if its generator were isotropic, it would be orthogonal both to itself and to the other eigenspace, hence to all of $\mathbb{C}^3$. Its orthogonal complement, which is the two-dimensional $-\lambda$ -eigenspace, is then nondegenerate as well. After choosing orthonormal bases in the two eigenspaces and adjusting one sign, the resulting orthogonal eigenbasis has determinant one. Thus A is diagonalized by an element of $SO_3(\mathbb{C})$, and every point is conjugate to

$$A_0 = \text{diag}(2\lambda, -\lambda, -\lambda).$$

The stabilizer preserves the one-dimensional 2λ -eigenspace and its orthogonal complement, hence is the normalizer $N(T)$ of a maximal torus.

For the scheme structure, let $B = (b_{ij})$ be a traceless symmetric tangent vector at A_0 . Linearizing $q = c$ and $Q = \lambda A$ gives

$$2 \operatorname{tr}(A_0 B) = 0$$

and

$$A_0 B + B A_0 - \frac{2}{3} \operatorname{tr}(A_0 B) I_3 = \lambda B.$$

A direct entrywise calculation gives

$$b_{11} = b_{22} = b_{33} = b_{23} = 0,$$

while b_{12} and b_{13} are free. Thus the tangent space has dimension 2, equal to

$$\dim G - \dim N(T) = 2.$$

The closed points form the two-dimensional orbit $G/N(T)$. Since $Z_{c,\lambda}$ is of finite type over $\mathbb{C}$, these closed points determine its support; in particular, every local ring at a closed point has Krull dimension 2. The G -action is transitive on the closed points, so the tangent-space calculation at A_0 holds at each of them. Every such local ring is therefore regular. Over the perfect field $\mathbb{C}$, this says that all closed points are smooth. The nonsmooth locus is closed and, if nonempty, would contain a closed point; hence $Z_{c,\lambda}$ is smooth and therefore reduced. It now has the same vanishing ideal as the reduced orbit $G/N(T)$, which proves the asserted scheme-theoretic equality. $\square$

Theorem 4.2 (Weight-four core rigidity). *Let V be a simple self-contragredient VOA of CFT type and central charge one. Suppose $J \in V_4$ is a nondegenerate primary, its cyclic Virasoro module is*

$$U(\operatorname{Vir})J \cong X_2 = L(1, 4),$$

and

$$\dim P_4(V) = 1.$$

Then

$$\langle L(1, 0), J \rangle_{\operatorname{VOA}} \cong M(1)^+.$$

Proof. Let $U_4 = \langle L(1, 0), J \rangle$. By Theorem 3.7, $U_4 \in \operatorname{Ind}_{\operatorname{CMY}}(\mathcal{C}_{\operatorname{sq}})$. Transport it to a commutative G -algebra B . It is generated by $E = W_4$ by Lemma 3.8.

Since $(U_4)_0 = \mathbb{C}\mathbf{1}$, we have $B^G = \mathbb{C}$. Moreover E occurs with multiplicity one: a copy of E corresponds to a copy of X_2 , and the highest-weight line of each copy lies in $P_4(V)$,

which is one-dimensional. Let $j : E \hookrightarrow B$ denote the generating copy. On the W_0 and W_4 summands, the multiplication map

$$\mathrm{Sym}^2 E \longrightarrow B$$

is therefore a scalar multiple of the unique equivariant map in each case; denote the scalar on the W_4 component by λ .

Since B is generated by E , the map

$$j \circ \iota_E : E^* \longrightarrow B$$

extends to a surjective G -algebra homomorphism

$$\pi : \mathrm{Sym}(E^*) \twoheadrightarrow B.$$

Since q is invariant, $\pi(q) = c\mathbf{1}$ for some $c \in \mathbb{C}$. The unit component is nonzero: Lemma 3.4 gives a nonzero vacuum coefficient for $Y(J, z)J$ because $(J, J) \neq 0$. Under ι_E , the polynomial q corresponds to an inverse Casimir tensor for the trace pairing, so contraction with this nonzero unit component remains nonzero. Hence $c \neq 0$. The two component identities say that $q - c$ and every coordinate function of the polynomial map $Q - \lambda \mathrm{id}_E$ lie in $\ker \pi$. Let $I_{c,\lambda} \subset \mathrm{Sym}(E^*)$ be the ideal generated by these polynomials. Thus π factors through a surjection

$$R_{c,\lambda} := \frac{\mathrm{Sym}(E^*)}{I_{c,\lambda}} \longrightarrow B.$$

The map is unital, so $R_{c,\lambda}$ is a nonzero finitely generated $\mathbb{C}$ -algebra. Its spectrum has a closed point by the Nullstellensatz. Lemma 4.1 therefore gives

$$R_{c,\lambda} \cong \mathcal{O}(G/N(T)).$$

By Lemma 2.15, this coordinate algebra has no nonzero proper G -stable ideal, so the kernel of the displayed surjection is zero. Thus

$$B \cong \mathcal{O}(G/N(T)).$$

Corollary 2.18 yields

$$U_4 \cong V_{A_1}^{N(T)}.$$

The torus-fixed subalgebra of V_{A_1} is the charge-zero Heisenberg VOA $M(1)$. The nontrivial Weyl element in $N(T)/T$ acts on the Heisenberg generator by sign, so the $N(T)$ -fixed subalgebra is exactly $M(1)^+$. Hence $U_4 \cong M(1)^+$. $\square$

4.2. Extracting the honest weight-four module in the lattice-orbifold character.

Let

$$L = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = 2k, \quad k \in \mathbb{Z}_{>0}.$$

The standard decomposition is

$$V_L^+ = M(1)^+ \oplus \bigoplus_{r \geq 1} M(1, r\alpha).$$

The zero-weight calculation in the proof of Lemma 3.9 gives, as a Virasoro module,

$$M(1)^+ \cong \bigoplus_{j \geq 0} X_{2j}.$$

The first lattice sector starts at conformal weight k .

Lemma 4.3 (Intervening-lattice saturation). *Assume $k > 4$ and*

$$\text{ch } V = \text{ch } V_L^+$$

for a simple self-contragredient CFT-type VOA V of central charge one. Then the one-dimensional primary space $P_4(V) = \mathbb{C}J$ is nondegenerate, and J generates

$$U(\text{Vir})J \cong X_2 = L(1, 4).$$

Proof. Because $k > 4$, the character agrees with the vacuum Virasoro character below weight four and has exactly one additional dimension at weight four. Lemma 3.3 gives a nondegenerate primary complement $\mathbb{C}J$ to $L(1, 0)_4$. The simple vacuum module $L(1, 0)$ has no positive-weight primary vectors, so in fact

$$P_4(V) = \mathbb{C}J.$$

The first singular vector of $M_{\text{Vir}}(1, 4)$ occurs at level five, that is, at absolute weight nine. Thus the cyclic module generated by J agrees with the Verma module below weight nine.

If $k > 9$, there is no lattice-sector contribution by weight nine, so Lemma 3.6 applies with $\varepsilon = 0$. If $k = 9$, the first lattice top contributes exactly one dimension and

$$\dim V_9 = \dim L(1, 0)_9 + \dim(X_2)_9 + 1;$$

the same lemma applies with $\varepsilon = 1$. Thus the first singular vector vanishes whenever $k \geq 9$.

Finally assume $5 \leq k \leq 8$, and write $C = U(\text{Vir})J$. The module C has no singular relation below weight nine. Since C is supported in weights at least four, its intersection with the simple vacuum module is zero. The two modules are also orthogonal. Indeed, for a descendant $xJ \in C$ and a vacuum descendant w of the same weight, contravariance moves

the negative Virasoro modes in x to positive modes acting on w and reduces the pairing to (J, w') for some $w' \in L(1, 0)_4$; this vanishes by the choice of J . Put

$$A = L(1, 0) \oplus C$$

as an orthogonal direct sum. Through weight $k < 9$, the module C has the same graded dimensions as X_2 . Moreover, $M(1)^+ = X_0 \oplus X_2 \oplus X_4 \oplus \cdots$, and the next summand X_4 starts only in weight 16. In the model character the first lattice sector contributes one top vector in weight k , while every sector of charge at least two starts in weight at least $4k$. It follows that

$$V_n = A_n \quad (n < k), \quad \dim V_k = \dim A_k + 1.$$

Below the first singular level, the contravariant form on C is a nonzero multiple of the Shapovalov form and is nondegenerate. The restriction to the vacuum module is nondegenerate as well, so A_k is nondegenerate. Its one-dimensional orthogonal complement is therefore nondegenerate, and we may choose a nonisotropic vector

$$F \in A_k^\perp.$$

For $n < k$, the model character is already exhausted by A_n . If $1 \leq r \leq k$, then $L(r)F \in V_{k-r} = A_{k-r}$, and for every $a \in A_{k-r}$ invariance gives

$$(L(r)F, a) = (F, L(-r)a) = 0.$$

The form on A_{k-r} is nondegenerate, so $L(r)F = 0$; for $r > k$ the target has negative weight. Hence F is primary. Since none of 5, 6, 7, 8 is of the form $n^2/4$ with $n \in \mathbb{Z}_{\geq 0}$, the $c = 1$ Verma module $M_{\text{Vir}}(1, k)$ is irreducible, and therefore

$$U(\text{Vir})F \cong L(1, k), \quad \sum_{n \geq 0} \dim L(1, k)_{k+n} q^{k+n} = q^k \prod_{j \geq 1} (1 - q^j)^{-1}.$$

The module $D = U(\text{Vir})F$ is orthogonal to A : for a Virasoro descendant xF and $a \in A$, contravariance moves x to an element of the positive enveloping algebra acting on a , and the resulting weight- k vector lies in A_k , which is orthogonal to F . Since $D \cong L(1, k)$ is irreducible and $(F, F) \neq 0$, the restricted form on D is nondegenerate. It follows that $D \cap A = 0$.

No second lattice sector appears by weight nine because $4k > 9$. The character formula therefore gives

$$\dim V_9 = \dim L(1, 0)_9 + \dim(X_2)_9 + \dim L(1, k)_9.$$

If the first singular vector in the J -cyclic module survived, the minimal-kernel argument from Lemma 3.6 would make the Verma map injective through weight nine. Hence

$$\dim C_9 = \dim(X_2)_9 + 1.$$

We have $A = L(1, 0) \oplus C$ and $D \cap A = 0$. Hence

$$L(1, 0)_9 \oplus C_9 \oplus D_9 \subseteq V_9,$$

and consequently

$$\dim V_9 \geq \dim L(1, 0)_9 + \dim(X_2)_9 + 1 + \dim L(1, k)_9,$$

contrary to the preceding equality. Hence the singular vector vanishes in every case, and $U(\text{Vir})J \cong X_2$. $\square$

Corollary 4.4. *Under the hypotheses of Lemma 4.3,*

$$\langle L(1, 0), J \rangle \cong M(1)^+.$$

Proof. Lemma 4.3 gives $P_4(V) = \mathbb{C}J$ and

$$U(\text{Vir})J \cong X_2.$$

Theorem 4.2 now applies. $\square$

5. THE O_2 CHARGE CATEGORY AND THE RANK-ONE ORBIFOLD BRANCH

5.1. Compact-lattice normalization of the charge category. Fix $k \in \mathbb{Z}_{>0}$ and choose a one-dimensional Heisenberg charge λ satisfying

$$\langle \lambda, \lambda \rangle = 2k.$$

Let

$$L_k = \mathbb{Z}\lambda, \quad T_k = \text{Hom}(L_k, U(1)) \cong U(1).$$

The compact torus T_k acts on the lattice VOA V_{L_k} by the charge characters.

Theorem 5.1 (The integral Fock category). *Let*

$$\mathcal{P}_k = \langle \mathcal{F}_{r\lambda} \mid r \in \mathbb{Z} \rangle_{\oplus}$$

be the full subcategory of finite direct sums of the indicated Heisenberg Fock modules. Then

$$V_{L_k}^{T_k} = M(1), \quad \mathcal{P}_k \simeq_{\text{sym}} \text{Rep } \mathbb{G}_m.$$

The Fock module $\mathcal{F}_{r\lambda}$ corresponds to the character $z \mapsto z^r$.

Proof. The lattice VOA has the charge decomposition

$$V_{L_k} = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}_{r\lambda}.$$

Distinct charge sectors carry distinct characters of T_k , so the fixed-point algebra is the charge-zero sector $\mathcal{F}_0 = M(1)$. The hypotheses of McRae's compact-orbifold theorem hold here. The lattice algebra V_{L_k} is a simple ordinary VOA, the group T_k is compact, and its action is continuous on every finite-dimensional homogeneous subspace, since that subspace is a finite sum of charge spaces. The ambient Heisenberg category of finite direct sums of Fock modules is a vertex tensor category and contains every multiplicity module $\mathcal{F}_{r\lambda}$; this is the setting recalled in [19, Example 4.11]. Thus Assumption 3.1 and the tensor-category hypothesis of [19, Corollary 4.8] are satisfied. Remark 4.9 of that paper also shows that the tensor products in the resulting subcategory do not depend on the chosen ambient module category. The argument uses only the charge lattice, and therefore applies to every even rank-one lattice L_k .

In McRae's notation an ordinary VOA has trivial abelian grading group, so the modifying cocycle is $(F, \Omega) = (1, 1)$. The resulting category is symmetric and equivalent to $\text{Rep } T_k$. Finally, finite-dimensional complex representations of $U(1)$ and rational representations of its complexification $\mathbb{G}_m$ have the same character lattice $\mathbb{Z}$. This identifies the charge- r Fock module with the character $z \mapsto z^r$. $\square$

The lattice inversion $\theta : \lambda \mapsto -\lambda$ lifts to an involution of V_{L_k} , normalizes T_k , and sends $\mathcal{F}_{r\lambda}$ to $\mathcal{F}_{-r\lambda}$. Together, T_k and θ generate the compact group

$$K_k := T_k \rtimes \langle \theta \rangle \cong O(2).$$

Theorem 5.2 (The integral $M(1)^+$ charge category). *In the C_1 -cofinite $M(1)^+$ tensor category, set*

$$\mathbf{1} = M(1)^+, \quad \varepsilon = M(1)^-, \quad Y_r = M(1, r\lambda) \quad (r \geq 1),$$

and let $\mathcal{D}_k$ be the full subcategory of finite direct sums of these simple modules. Then there is a symmetric vertex tensor equivalence

$$\Psi_k : \text{Rep } O_2(\mathbb{C}) \xrightarrow{\sim} \mathcal{D}_k.$$

Under the equivalence,

$$\mathbf{1} \longmapsto \mathbf{1}, \quad \det \longmapsto \varepsilon, \quad \rho_r \longmapsto Y_r,$$

where ρ_r is the two-dimensional irreducible $O_2(\mathbb{C})$ -representation with torus weights $\{r, -r\}$.

Proof. Adamović–Lin–Yang prove that the category $\mathcal{C}_1(M(1)^+)$ of grading-restricted C_1 -cofinite modules is a vertex and braided tensor category [2, Theorem 4.3]. By [2, Corollary 3.4 and Theorem 3.5], it contains $M(1)^\pm$ and every $M(1, \mu)$ with $\mu \neq 0$. Thus all the modules defining $\mathcal{D}_k$ lie in one vertex tensor category. The ambient category is not semisimple; only the subcategory singled out below will be.

The element θ acts on T_k by inversion. The group K_k acts by genuine automorphisms of V_{L_k} , and

$$V_{L_k}^{K_k} = (V_{L_k}^{T_k})^{(\theta)} = M(1)^+.$$

As a $K_k \times M(1)^+$ -module, the lattice VOA has the Schur–Weyl decomposition

$$V_{L_k} \cong \mathbf{1} \otimes M(1)^+ \oplus \det \otimes M(1)^- \oplus \bigoplus_{r \geq 1} \rho_r^{\text{cpt}} \otimes M(1, r\lambda),$$

where ρ_r^{cpt} is the two-dimensional representation with T_k -weights $\{r, -r\}$. Here V_{L_k} is again simple and ordinary, K_k is compact, and the action is continuous on every homogeneous subspace. The multiplicity modules in the displayed decomposition all belong to the vertex tensor category $\mathcal{C}_1(M(1)^+)$ established in [2, Theorem 4.3]. Consequently [19, Corollary 4.8 and Remark 4.10] applies at the level of the full $P(z)$ -tensor structures. Since the abelian grading group of the ordinary VOA V_{L_k} is trivial, the modifying cocycle is trivial and the source is the ordinary symmetric category $\text{Rep } K_k$. By [19, Remark 4.9], its image is a tensor subcategory independent of the ambient choice; here it is precisely $\mathcal{D}_k$, which is therefore semisimple.

Complexification identifies finite-dimensional continuous representations of $K_k \cong O(2)$ with rational representations of $O_2(\mathbb{C})$. Under this identification, the three types of multiplicity spaces correspond respectively to $\mathbf{1}$, $\det$, and ρ_r . This proves the claimed symmetric equivalence. The resulting fusion decompositions coincide with [2, Theorem 4.6]. $\square$

The fusion formulas in this subcategory are therefore

$$\varepsilon \boxtimes \varepsilon \cong \mathbf{1}, \quad \varepsilon \boxtimes Y_r \cong Y_r,$$

and

$$Y_r \boxtimes Y_s \cong \begin{cases} Y_{r+s} \oplus Y_{|r-s|}, & r \neq s, \\ Y_{2r} \oplus \mathbf{1} \oplus \varepsilon, & r = s. \end{cases}$$

These are also exactly the formulas of [2, Theorem 4.6].

Lemma 5.3 (Ambient subquotients of $\mathcal{D}_k$). *The full subcategory $\mathcal{D}_k$ is closed under ambient $M(1)^+$ -submodules and quotients, and its inclusion in the grading-restricted $M(1)^+$ -module category is exact.*

Proof. Every object of $\mathcal{D}_k$ is, as an actual $M(1)^+$ -module, a finite direct sum of the simple modules $M(1)^+$, $M(1)^-$, and $M(1, r\lambda)$. It is therefore semisimple in the ordinary module-theoretic sense. Every ambient submodule is a direct summand and hence a finite direct sum of the same simple modules; the same holds for quotients. Kernels and cokernels in the full subcategory consequently agree with the ambient ones. $\square$

Proposition 5.4 (Direct-limit certificate for $\mathcal{D}_k$). *The category $\mathcal{D}_k$ satisfies the hypotheses of the Creutzig–McRae–Yang direct-limit completion theorem. Hence $\text{Ind}_{\text{CMY}}(\mathcal{D}_k)$ carries the vertex and braided tensor structures of [6, Theorem 1.1]. The extension–algebra correspondence of [6, Theorem 1.3] applies to commutative algebra objects that are integrally $L(0)$ -graded and satisfy the required grading restrictions; in particular, it applies to the extension-admissible objects of Definition 2.12. Moreover, if a grading-restricted VOA extension $A \supset M(1)^+$ contains a finite direct sum S of simple objects of $\mathcal{D}_k$, then*

$$\langle M(1)^+, S \rangle_{\text{VOA}} \in \text{Ind}_{\text{CMY}}(\mathcal{D}_k).$$

Proof. The base VOA $M(1)^+$ is the tensor unit of $\mathcal{D}_k$. Lemma 5.3 verifies closure under actual ambient submodules and quotients, and finite direct sums are immediate. Its objects are finite direct sums of finitely generated simple modules. The $P(z)$ -tensor products and parallel transports are the restrictions of those in the full $M(1)^+$ category by the $P(z)$ -compatibility in [19, Corollary 4.8 and Remark 4.10]. These observations verify conditions (1)–(4) of [6, Theorem 1.1].

We prove a coefficient-image statement stronger than condition (5). Let $M, N \in \mathcal{D}_k$, let X be an arbitrary generalized $M(1)^+$ -module, and let $\mathcal{Y}$ be an intertwining operator of type $\left(\begin{smallmatrix} X \\ M \ N \end{smallmatrix}\right)$. The commutator formula shows that its coefficient span I is an $M(1)^+$ -submodule of X . Corestriction gives an intertwining operator

$$\mathcal{Y}^I : M \otimes N \longrightarrow I\{z\}[\log z]$$

whose coefficients span I . Since M and N are C_1 -cofinite, Corollary 2.14 of [6] applies with the arbitrary generalized target X and shows that I is grading restricted and C_1 -cofinite. Hence $I \in \mathcal{C}_1(M(1)^+)$ by [2, Theorem 4.2]. The ambient vertex tensor category of [2,

Theorem 4.3] therefore supplies the $P(z)$ -tensor universal property and a morphism

$$f : M \boxtimes N \longrightarrow I.$$

The inverse formula [19, Equation (2.1)] lifts the equality of the associated $P(z)$ -intertwining maps to the formal intertwining operators. Every coefficient of $\mathcal{Y}^I$ therefore lies in $\text{Im } f$. These coefficients span I , so f is an epimorphism. The fusion formulas above make $M \boxtimes N$ a finite semisimple $\mathcal{D}_k$ -object. Since f is an ambient epimorphism, Lemma 5.3 shows that its quotient I belongs to $\mathcal{D}_k$.

Taking $X \in \text{Ind}_{\text{CMY}}(\mathcal{D}_k)$ proves condition (5) of [6, Theorem 1.1]. That theorem supplies the direct-limit tensor structures. The extension statement, with the integrality and grading restrictions in the proposition, follows from [6, Theorem 1.3].

For the final assertion, put

$$E_0 = M(1)^+ + S \subset A.$$

It is an ambient quotient of the finite semisimple object $M(1)^+ \oplus S$, so Lemma 5.3 gives $E_0 \in \mathcal{D}_k$. Suppose that $E_r \in \mathcal{D}_k$ and choose an internal decomposition

$$E_r = \bigoplus_{\lambda \in \Lambda_r} T_\lambda$$

into its actual embedded simple summands. For every ordered pair λ, μ , restrict the ambient vertex operator of A to $T_\lambda \otimes T_\mu$. The VOA A is a generalized $M(1)^+$ -module, so the stronger coefficient-image statement just proved puts the image $I_{\lambda, \mu}$ in $\mathcal{D}_k$. Define

$$E_{r+1} = E_r + \sum_{\lambda, \mu \in \Lambda_r} I_{\lambda, \mu}.$$

This is an ambient quotient of a finite direct sum of semisimple $\mathcal{D}_k$ -objects, and Lemma 5.3 again puts it in $\mathcal{D}_k$. Induction on the depth of a bracketed mode word gives

$$\langle M(1)^+, S \rangle_{\text{VOA}} = \bigcup_{r \geq 0} E_r.$$

The right-hand side is a directed union of $\mathcal{D}_k$ -subobjects, proving the asserted membership in $\text{Ind}_{\text{CMY}}(\mathcal{D}_k)$ without assuming it for the ambient extension. $\square$

Lemma 2.10 now extends Ψ_k to an exact symmetric monoidal equivalence

$$\widehat{\Psi}_k : \text{Rep}^{\text{rat}} O_2(\mathbb{C}) \xrightarrow{\sim} \text{Ind}_{\text{CMY}}(\mathcal{D}_k).$$

5.2. The first lattice module.

Lemma 5.5 (Lowest-line criterion for $M(1)^+$). *Let M be a grading-restricted $M(1)^+$ -module and suppose that, for some conformal weight h ,*

$$M_h = \mathbb{C}v, \quad M_{h-r} = 0 \quad (r \in \mathbb{Z}_{>0}).$$

Then the cyclic module $M(1)^+ \cdot v$ is a highest-weight $M(1)^+$ -module in the sense of [2, Section 3.4].

Proof. For homogeneous $a \in M(1)^+$ and $n \geq \text{wt}(a)$,

$$a_n v \in M_{h+\text{wt}(a)-n-1} = 0.$$

The zero mode $o(a) = a_{\text{wt}(a)-1}$ preserves the one-dimensional lowest space $\mathbb{C}v$ and therefore acts there by a scalar. Thus $\mathbb{C}v$ is a one-dimensional module for the Zhu algebra and v satisfies the highest-weight conditions used in [2, Section 3.4]. By definition, the module under consideration is the cyclic module generated by this highest-weight vector. $\square$

Lemma 5.6 (Orthogonal first-lattice extraction). *Assume $k > 4$ and*

$$\text{ch } V = \text{ch } V_L^+, \quad L = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = 2k.$$

Let

$$U \cong M(1)^+$$

be the subVOA obtained in Corollary 4.4, and put $K = U^\perp$ with respect to the invariant bilinear form. Then:

- (i) *K is a grading-restricted U -module and its homogeneous bilinear form is nondegenerate;*
- (ii) *$K_j = 0$ for $j < k$ and $\dim K_k = 1$;*
- (iii) *every $0 \neq F \in K_k$ is nonisotropic and is a U -highest-weight vector;*
- (iv) *the U -module generated by F is irreducible and*

$$U \cdot F \cong M(1, \lambda), \quad \langle \lambda, \lambda \rangle = 2k.$$

Proof. The restriction of the invariant form to the simple subVOA U is nondegenerate: its radical is an ideal, and the vacuum has nonzero norm. Thus

$$V = U \oplus U^\perp$$

orthogonally in each weight. In particular, $K = U^\perp$ is graded, lower bounded, and has finite-dimensional homogeneous spaces. To verify that it is a U -module, let $a, u' \in U$ and $v \in K$. Invariance of the bilinear form gives

$$(Y(a, z)v, u') = (v, Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u').$$

Because U is a conformal subVOA, every coefficient on the right belongs to U , and the right-hand side vanishes. Comparing coefficients shows that $a_nv \in K$ for every $n \in \mathbb{Z}$. Hence K is a grading-restricted U -module; its homogeneous form is nondegenerate by the orthogonal decomposition above.

The character of V_L^+ is

$$\text{ch } M(1)^+ + \sum_{r \geq 1} \text{ch } M(1, r\alpha),$$

and the first summand outside $M(1)^+$ begins at weight k with one-dimensional top. Character equality therefore gives $K_j = 0$ for $j < k$ and $\dim K_k = 1$. The form on K_k is nondegenerate, so every nonzero $F \in K_k$ is nonisotropic. If $a \in U$ is homogeneous and $n \geq \text{wt}(a)$, then

$$a_n F \in K_{k+\text{wt}(a)-n-1} = 0.$$

The zero mode $o(a) = a_{\text{wt}(a)-1}$ preserves the one-dimensional space K_k and hence acts by a scalar. Lemma 5.5 shows that $U \cdot F$ is a grading-restricted highest-weight module in the precise sense of [2, Section 3.4].

By [2, Theorem 3.8], an $M(1)^+$ highest-weight module of nonzero conformal weight is irreducible. The classification recalled in [2, Section 2.1] says that the non-Fock irreducibles have lowest conformal weights 0, 1, 1/16, or 9/16. Since the present lowest weight is the integer $k > 4$, the generated module must be of Fock type:

$$U \cdot F \cong M(1, \lambda)$$

with $\langle \lambda, \lambda \rangle / 2 = k$; the sign of λ is immaterial because $M(1, \lambda) \cong M(1, -\lambda)$. The radical of the restriction of the invariant form to $U \cdot F$ is a U -submodule. Since $U \cdot F$ is irreducible and $(F, F) \neq 0$, this radical is zero. Thus the generated Fock module is nondegenerate. $\square$

5.3. The nonzero quadratic orbit. Let ρ_1 be the standard two-dimensional representation of $O_2(\mathbb{C})$. Fix a nondegenerate invariant symmetric bilinear form

$$\beta_Q : \rho_1 \otimes \rho_1 \longrightarrow \mathbf{1}$$

and let $Q_\rho \in \text{Sym}^2(\rho_1^*)^{O_2}$ be the corresponding quadratic tensor. Write $\iota_\rho : \rho_1 \xrightarrow{\sim} \rho_1^*$ for the induced equivariant self-duality. Its inverse Casimir tensor is

$$\check{Q}_\rho = (\iota_\rho^{-1})^{\otimes 2}(Q_\rho) \in \text{Sym}^2(\rho_1)^{O_2}.$$

By construction,

$$\beta_Q(\check{Q}_\rho) = \dim \rho_1 = 2.$$

We retain this distinction between the quadratic coordinate tensor Q_ρ and the invariant tensor $\check{Q}_\rho$.

Lemma 5.7 (The O_2 quadric). *For $c \in \mathbb{C}^\times$, set*

$$R_c = \frac{\text{Sym}(\rho_1^*)}{(Q_\rho - c)}.$$

Then

$$\text{Spec } R_c \cong O_2(\mathbb{C})/H_{\text{refl}},$$

where $H_{\text{refl}} \cong \mathbb{Z}_2$ is the stabilizer of a nonzero nonisotropic vector. In particular R_c is reduced and has no nonzero proper O_2 -stable ideals. As an O_2 -module,

$$R_c \cong \mathbf{1} \oplus \bigoplus_{r \geq 1} \rho_r.$$

Proof. The spectrum is the nonzero norm quadric

$$\{x \in \rho_1 : Q_\rho(x) = c\}.$$

The complex orthogonal group acts transitively on this quadric. The stabilizer of a nonzero nonisotropic vector fixes its orthogonal line and is the order-two reflection subgroup. Thus the quadric is the homogeneous space O_2/H_{refl} ; it is smooth and reduced. Lemma 2.15 applied to O_2 gives the ideal statement.

Peter–Weyl for the homogeneous space gives

$$\mathcal{O}(O_2/H_{\text{refl}}) \cong \bigoplus_{S \in \text{Irr}(O_2)} S \otimes (S^*)^{H_{\text{refl}}}.$$

The trivial representation and each ρ_r have one-dimensional reflection-fixed subspace, while $\det$ has none. This gives the stated decomposition. $\square$

Lemma 5.8 (Admissibility of the O_2 quadric). *The algebra R_c is extension-admissible under Theorem 5.2. Its underlying $M(1)^+$ -module is*

$$M(1)^+ \oplus \bigoplus_{r \geq 1} M(1, r\lambda).$$

Proof. The decomposition in Lemma 5.7 contains one copy of the tensor unit and no other trivial summand. Hence the unit map is injective and $\text{Hom}(\mathbf{1}, R_c) = \mathbb{C}$. The highest conformal weights are 0 and

$$\frac{\langle r\lambda, r\lambda \rangle}{2} = kr^2 \in \mathbb{Z}_{>0},$$

and each Fock module has spectrum $kr^2 + \mathbb{Z}_{\geq 0}$. Thus $L(0)$ is semisimple, the whole object is lower bounded, and all its conformal weights are nonnegative integers; in particular, the twist is the identity. At any fixed weight N , only finitely many r satisfy $kr^2 \leq N$, and each Fock module has finite-dimensional homogeneous spaces. Thus all conditions in Definition 2.12 hold. $\square$

Lemma 5.9 (The standard rank-one orbifold algebra). *Let $k \in \mathbb{Z}_{>0}$ and let $L = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 2k$. The extension*

$$V_L^+ \supset M(1)^+$$

belongs to $\text{Ind}_{\text{CMY}}(\mathcal{D}_k)$ and is generated over $M(1)^+$ by its first lattice module $M(1, \alpha)$. If B_k^{std} is the corresponding commutative rational $O_2(\mathbb{C})$ -algebra, then

$$B_k^{\text{std}} \cong R_{c_k} = \frac{\text{Sym}(\rho_1^*)}{(Q_\rho - c_k)}$$

for some $c_k \in \mathbb{C}^\times$.

Proof. The argument uses only $k > 0$. The standard decomposition

$$V_L^+ = M(1)^+ \oplus \bigoplus_{r \geq 1} M(1, r\alpha)$$

expresses V_L^+ as the directed union of its finite partial sums. Consequently,

$$V_L^+ \in \text{Ind}_{\text{CMY}}(\mathcal{D}_k).$$

Let θ be the lift of lattice inversion and put

$$F_r = e^{r\alpha} + \theta(e^{r\alpha}) \quad (r \geq 1).$$

The lattice vertex-operator formula gives

$$Y(e^\alpha, z)e^{r\alpha} = \varepsilon(\alpha, r\alpha)z^{2kr}(e^{(r+1)\alpha} + O(z)), \quad \varepsilon(\alpha, r\alpha) \neq 0.$$

Let p_s denote projection to the sum of the two charge spaces $\mathcal{F}_{s\alpha} \oplus \mathcal{F}_{-s\alpha}$. Since F_1 and F_r are fixed by θ , so is $(F_1)_{-2kr-1}F_r$. Its $(r+1)\alpha$ -component is $\varepsilon(\alpha, r\alpha)e^{(r+1)\alpha} \neq 0$, and θ -equivariance forces the $-(r+1)\alpha$ -component to be its nonzero translate. Hence

$$p_{r+1}((F_1)_{-2kr-1}F_r) = c_r F_{r+1}, \quad c_r \neq 0.$$

The remaining components have charges $\pm(r-1)\alpha$; for $r=1$ they lie in $M(1)^+$. Once the sectors through charge r have been generated, these lower-charge terms may be subtracted. Induction therefore recovers F_{r+1} and then the simple module $M(1, (r+1)\alpha)$. Hence $M(1, \alpha)$ generates all of V_L^+ over $M(1)^+$.

Let B_k^{std} be the corresponding O_2 -algebra. The preceding generation shows that it is generated by ρ_1 , while $(V_L^+)_0 = \mathbb{C}\mathbf{1}$ gives $(B_k^{\text{std}})^{O_2} = \mathbb{C}\mathbf{1}$. The self-duality ι_ρ therefore yields a surjection

$$\pi_k : \text{Sym}(\rho_1^*) \twoheadrightarrow B_k^{\text{std}}, \quad \pi_k(Q_\rho) = c_k \mathbf{1}$$

for some $c_k \in \mathbb{C}$.

The restriction to V_L^+ of the standard lattice invariant form is nondegenerate. If S, T are two nonisomorphic irreducible summands in

$$V_L^+ = M(1)^+ \oplus \bigoplus_{r \geq 1} M(1, r\alpha),$$

invariance identifies their cross-pairing with an $M(1)^+$ -homomorphism $S \rightarrow T'$, which vanishes by Schur's lemma. Thus this multiplicity-free decomposition is orthogonal, and the global nondegeneracy implies nondegeneracy on each summand. In particular, a nonzero top vector $F_1 \in M(1, \alpha)$ satisfies $(F_1, F_1) \neq 0$. The unit component $\mu_0^{\text{std}} : \rho_1 \otimes \rho_1 \rightarrow \mathbf{1}$ of multiplication is therefore nonzero by Lemma 3.4. Since the relevant Hom-space is one-dimensional,

$$\mu_0^{\text{std}} = b_k \beta_Q \quad (b_k \in \mathbb{C}^\times).$$

Consequently

$$\pi_k(Q_\rho) = \mu_0^{\text{std}}(\check{Q}_\rho) \mathbf{1} = 2b_k \mathbf{1} \neq 0,$$

so $c_k \neq 0$. Hence π_k factors through a surjective O_2 -algebra map $R_{c_k} \twoheadrightarrow B_k^{\text{std}}$. Its kernel is zero by Lemma 5.7, proving the assertion. $\square$

Theorem 5.10 (Rank-one orbifold character rigidity for $k > 4$). *Let $L = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 2k$ and $k > 4$. Let V be a simple self-contragredient CFT-type VOA of central charge one and assume only*

$$\text{ch } V = \text{ch } V_L^+.$$

Then

$$V \cong V_L^+.$$

Proof. By Corollary 4.4, V contains a subVOA

$$U \cong M(1)^+.$$

Lemma 5.6 gives a nondegenerate irreducible U -submodule

$$Y_1 \cong M(1, \lambda), \quad \langle \lambda, \lambda \rangle = 2k.$$

Set

$$W = \langle U, Y_1 \rangle_{\text{VOA}}.$$

By Proposition 5.4, $W \in \text{Ind}_{\text{CMY}}(\mathcal{D}_k)$. Put

$$B = \widehat{\Psi}_k^{-1}(W).$$

Lemma 3.8, applied to Ψ_k , shows that the commutative rational O_2 -algebra B is generated by the standard representation ρ_1 .

Under the equivalence, $B^{O_2} = \text{Hom}_{O_2}(\mathbf{1}, B)$ identifies with $\text{Hom}_{M(1)^+}(M(1)^+, W)$. A module map is determined by the image of the vacuum, and $W_0 = \mathbb{C}\mathbf{1}$ because $W \subset V$ and V is of CFT type. Hence $B^{O_2} = \mathbb{C}\mathbf{1}$. Commutativity of B in the symmetric category makes the multiplication on $\rho_1 \otimes \rho_1$ factor through $\text{Sym}^2 \rho_1$ and kill $\bigwedge^2 \rho_1 = \det$. Let $j : \rho_1 \hookrightarrow B$ be the generating copy. The map

$$j \circ \iota_\rho^{-1} : \rho_1^* \longrightarrow B$$

extends uniquely to a surjective unital O_2 -algebra morphism

$$\pi_B : \text{Sym}(\rho_1^*) \twoheadrightarrow B.$$

Since Q_ρ is invariant, $\pi_B(Q_\rho) = c\mathbf{1}$ for some $c \in \mathbb{C}$.

In fact, $c \neq 0$. Let

$$\mu_0 : \rho_1 \otimes \rho_1 \longrightarrow \mathbf{1}$$

be the unit component of the multiplication. Under Ψ_k , it is the unit component of $Y_1 \boxtimes Y_1 \rightarrow W$. Every nonunit simple in $\mathcal{D}_k$ has positive lowest conformal weight, so the weight-zero coefficient of the product of two lowest-weight vectors lies entirely in this channel. For the nonisotropic top vector $F \in Y_1$, Lemma 3.4 gives

$$F_{2k-1}F = (-1)^k(F, F)\mathbf{1} \neq 0.$$

Thus $\mu_0 \neq 0$. The space $\text{Hom}_{O_2}(\rho_1 \otimes \rho_1, \mathbf{1})$ is one-dimensional, so

$$\mu_0 = b\beta_Q \quad (b \in \mathbb{C}^\times).$$

By the normalization of the inverse Casimir tensor,

$$\pi_B(Q_\rho) = \mu_0(\check{Q}_\rho)\mathbf{1} = 2b\mathbf{1} \neq 0.$$

This proves $c \in \mathbb{C}^\times$. The map π_B therefore factors through a surjective O_2 -algebra homomorphism

$$R_c = \text{Sym}(\rho_1^*)/(Q_\rho - c) \twoheadrightarrow B.$$

By Lemma 5.7, R_c has no nonzero proper O_2 -stable ideal, so the kernel vanishes. Thus

$$B \cong R_c.$$

Lemma 5.8 makes the extension correspondence applicable.

By Lemma 5.9, the standard extension $V_L^+ \supset M(1)^+$ corresponds to $B_k^{\text{std}} \cong R_{c_k}$ for some $c_k \in \mathbb{C}^\times$.

Choose $a \in \mathbb{C}^\times$ with $a^2 c = c_k$. Sending the generating copy of ρ_1 in R_{c_k} to a times the generating copy in R_c defines an O_2 -algebra isomorphism $R_{c_k} \rightarrow R_c$. After applying $\widehat{\Psi}_k$, this is an isomorphism of commutative algebra objects in $\text{Ind}_{\text{CMY}}(\mathcal{D}_k)$. Both algebras are extension-admissible by Lemma 5.8, so [6, Theorem 1.3] identifies it with an isomorphism of VOA extensions of the fixed base $M(1)^+$. In particular, the base conformal vector and the $L(0)$ -grading are preserved, and

$$W \cong V_L^+, \quad \text{ch } W = \text{ch } V_L^+ = \text{ch } V.$$

Since $W_n \subset V_n$ for every n , equality of all homogeneous dimensions yields $W_n = V_n$ and hence $W = V$. $\square$

6. THE FOUR LOW-NORM LATTICE-ORBIFOLD CASES

Throughout this section, V is assumed to satisfy the hypotheses of Theorem 1.1.

6.1. A uniform low-weight character computation. For a central-charge-one VOA W with nonnegative integral grading, write

$$\text{grch } W := q^{1/24} \text{ch } W = \sum_{n \geq 0} (\dim W_n) q^n.$$

Thus grch is the ordinary graded dimension, with the common factor $q^{-1/24}$ removed. Put

$$P(q) = \prod_{n \geq 1} (1 - q^n)^{-1}, \quad T(q) = \prod_{n \geq 1} (1 + q^n)^{-1}.$$

Lemma 6.1 (Rank-one orbifold character formula). *For $L = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 2k$,*

$$\text{grch } M(1)^+ = \frac{P(q) + T(q)}{2},$$

and

$$\text{grch } V_L^+ = \frac{P(q) + T(q)}{2} + P(q) \sum_{r \geq 1} q^{kr^2}.$$

Proof. The Heisenberg involution acts by -1 on every oscillator $\alpha(-n)$. Hence its graded trace on the oscillator algebra is

$$\prod_{n \geq 1} (1 + q^n)^{-1} = T(q),$$

whereas the ordinary graded trace is $P(q)$. Averaging the identity and involution traces gives the fixed-point graded dimension $(P + T)/2$.

For every $r \geq 1$, the pair of lattice charges $\pm r\alpha$ contributes one full oscillator Fock space to the $+$ -fixed subspace, with lowest conformal weight kr^2 . Its graded dimension is $q^{kr^2} P(q)$. Summing over r gives the formula. $\square$

Proposition 6.2 (Low-weight dimensions). *Let $\Lambda_k = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 2k$ and write*

$$\text{grch } V_{\Lambda_k}^+ = \sum_{n \geq 0} d_n^{(k)} q^n.$$

For $1 \leq k \leq 4$ and $0 \leq n \leq 9$,

$k \backslash n$	0	1	2	3	4	5	6	7	8	9
1	1	1	2	3	7	9	15	21	32	44
2	1	0	2	2	5	6	11	14	24	30
3	1	0	1	2	4	5	9	12	19	25
4	1	0	1	1	4	4	8	10	17	21

Proof. The expansions needed are

$$P(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + 30q^9 + O(q^{10}),$$

and

$$T(q) = 1 - q - q^3 + q^4 - q^5 + q^6 - q^7 + 2q^8 - 2q^9 + O(q^{10}).$$

Insert these in Lemma 6.1 and add the finitely many shifts $q^{kr^2} P(q)$ that can contribute through weight nine. Direct coefficient comparison gives the table. $\square$

6.2. The cases $k = 1$ and $k = 2$.

Theorem 6.3 (The case $k = 1$). *If*

$$\mathrm{ch} V = \mathrm{ch} V_{\Lambda_1}^+, \quad \langle \alpha, \alpha \rangle = 2,$$

then

$$V \cong V_{\Lambda_1}^+.$$

Proof. Proposition 6.2 gives $\dim V_1 = 1$. For a strongly rational VOA the Lie algebra V_1 is reductive, so its Lie rank is one. By Li's criterion [14, Corollary 3.2], simplicity, CFT type and self-contragredience imply $L(1)V_1 = 0$. Thus the standing hypotheses give strong CFT type in the convention of Dong–Mason, and Theorems 1–3 of [11] apply. They give

$$\mathrm{rank} V_1 \leq \tilde{c} = 1,$$

and equality $\mathrm{rank} V_1 = \tilde{c} = c = 1$ implies that V is a positive-definite rank-one lattice VOA V_K .

Dong–Griess prove the classical accidental isomorphism

$$V_{A_1}^+ \cong V_{2A_1},$$

where the generator of $2A_1$ has squared length 8 [7]. Therefore

$$\mathrm{ch} V_K = \mathrm{ch} V = \mathrm{ch} V_{A_1}^+ = \mathrm{ch} V_{2A_1}.$$

For a rank-one lattice VOA, the character is Θ_K/η , so $\Theta_K = \Theta_{2A_1}$. A positive-definite rank-one even lattice is determined by the squared length of its shortest positive generator, equivalently by the first nonzero exponent in its theta series. Hence $K \cong 2A_1$ and

$$V \cong V_{2A_1} \cong V_{A_1}^+ = V_{\Lambda_1}^+.$$

□

Theorem 6.4 (The case $k = 2$). *If*

$$\mathrm{ch} V = \mathrm{ch} V_{\Lambda_2}^+, \quad \langle \alpha, \alpha \rangle = 4,$$

then

$$V \cong V_{\Lambda_2}^+.$$

Proof. By Proposition 6.2,

$$V_1 = 0, \quad \dim V_2 = 2.$$

Theorem 4.7 of Zhang–Dong [22] applies to a rational, C_2 -cofinite VOA of moonshine type with $c = \tilde{c} = 1$ and $\dim V_2 = 2$. Here moonshine type follows from $V_0 = \mathbb{C}\mathbf{1}$ and $V_1 = 0$, and all the remaining hypotheses are part of the standing assumptions. The theorem therefore gives

$$V \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{1}{2}, 0\right).$$

For the rank-one lattice of squared norm four, Dong–Griess record the VOA isomorphism

$$V_{\Lambda_2}^+ \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{1}{2}, 0\right)$$

in [7, Remark 3.2]. This is an isomorphism of the entire fixed-point algebra, not merely an embedding furnished by two Ising vectors. Hence $V \cong V_{\Lambda_2}^+$. $\square$

6.3. The case $k = 3$.

Lemma 6.5. *Assume $\text{ch } V = \text{ch } V_{\Lambda_3}^+$. Then $P_3(V)$ is one-dimensional. Every nonzero $F \in P_3(V)$ satisfies*

$$(F, F) \neq 0, \quad U(\text{Vir})F \cong L(1, 3).$$

Proof. The table gives $\dim V_3 = 2$, whereas $\dim L(1, 0)_3 = 1$. The orthogonal complement of the vacuum Virasoro subspace is therefore a one-dimensional nondegenerate primary line, exactly as in Lemma 3.3.

At $c = 1$, a Verma module $M_{\text{Vir}}(1, h)$ is reducible precisely at the degenerate highest weights $h = n^2/4$ with $n \in \mathbb{Z}_{\geq 0}$ [15, 16]. Since 3 is not of this form, $M_{\text{Vir}}(1, 3)$ is irreducible. The nonzero map from $M_{\text{Vir}}(1, 3)$ induced by F is therefore injective, proving the claim. $\square$

Lemma 6.6. *Under the same assumptions, $P_4(V)$ is one-dimensional. Every nonzero $J \in P_4(V)$ satisfies*

$$(J, J) \neq 0, \quad U(\text{Vir})J \cong X_2 = L(1, 4).$$

Proof. Let

$$A = L(1, 0) \oplus U(\text{Vir})F,$$

where F is as in Lemma 6.5. The two summands are orthogonal: the invariant pairing would otherwise give a nonzero Virasoro homomorphism between the nonisomorphic simple modules $L(1, 3)$ and $L(1, 0)' \cong L(1, 0)$. The restriction of the form to $U(\text{Vir})F \cong L(1, 3)$

is nonzero because $(F, F) \neq 0$, and is therefore nondegenerate. Thus A is a nondegenerate Virasoro submodule.

At weight four, A contributes two vacuum descendants and the level-one descendant of F . Since $\dim V_4 = 4$, its orthogonal complement in V_4 is a one-dimensional nondegenerate line; choose $0 \neq J$ on this line. For $1 \leq n \leq 4$, the low-weight table gives $V_{4-n} = A_{4-n}$. If $a \in A_{4-n}$, invariance yields

$$(L(n)J, a) = (J, L(-n)a) = 0.$$

The form on $A_{4-n} = V_{4-n}$ is nondegenerate, so $L(n)J = 0$ for $1 \leq n \leq 4$; for $n > 4$ the same conclusion follows from the absence of negative conformal weights. Hence J is primary. Neither simple summand of A has a primary vector of weight four, and consequently $P_4(V) = \mathbb{C}J$.

By Proposition 3.5, the first singular vector of $M_{\text{Vir}}(1, 4)$ occurs at level five, hence at absolute weight nine, and it generates the maximal proper submodule. Thus the cyclic module generated by J agrees with the Verma module below weight nine. Suppose that the level-five singular vector survives and denote its image by $s \in V_9$. Then s is a primary of weight nine.

Let $C = U(\text{Vir})J$. The intersection of C with the semisimple module $L(1, 0) \oplus L(1, 3)$ is zero: any nonzero intersection would contain one of these two simple summands, but C has no vectors of conformal weight zero or three. If s survived, then at the first singular level C_9 would have the full Verma dimension $p(5) = 7$. Proposition 6.2 gives $\dim V_9 = 25$, whereas

$$\dim L(1, 0)_9 = 8, \quad \dim L(1, 3)_9 = p(6) = 11,$$

and therefore

$$\dim(L(1, 0) \oplus L(1, 3) \oplus C)_9 = 8 + 11 + 7 = 26,$$

a contradiction. Hence the singular vector is killed. The maximal submodule statement in Proposition 3.5 now shows that the cyclic module is X_2 . $\square$

Theorem 6.7 (The case $k = 3$). *If $\text{ch } V = \text{ch } V_{\Lambda_3}^+$, then*

$$V \cong V_{\Lambda_3}^+.$$

Proof. Lemma 6.6 and Theorem 4.2 give a subVOA

$$U \cong M(1)^+.$$

The invariant form restricts nondegenerately to the simple subVOA U : its radical is an ideal and the vacuum has nonzero norm. By the invariance calculation in Lemma 5.6, the orthogonal complement $K = U^\perp$ is a grading-restricted U -module, and the form on each of its homogeneous spaces is nondegenerate.

Character comparison gives $K_j = 0$ for $j < 3$ and $\dim K_3 = 1$. Thus every nonzero $F \in K_3$ is nonisotropic. If $a \in U$ is homogeneous, then

$$a_n F \in K_{3+\text{wt}(a)-n-1} = 0 \quad (n \geq \text{wt}(a)),$$

while the zero mode $o(a) = a_{\text{wt}(a)-1}$ acts by a scalar on the one-dimensional top K_3 . Lemma 5.5 shows that $U \cdot F$ is a grading-restricted highest-weight module in the precise sense of [2, Section 3.4]. Its conformal weight is $3 \neq 0$, so [2, Theorem 3.8] shows that it is irreducible. The classification recalled in [2, Section 2.1] lists the non-Fock irreducibles with lowest conformal weights 0, 1, $1/16$, and $9/16$. An irreducible module of lowest weight 3 must therefore be of Fock type. Hence, for some charge μ ,

$$U \cdot F \cong M(1, \mu), \quad \langle \mu, \mu \rangle = 6.$$

Let λ_0 be the charge of norm six used to define the fixed category $\mathcal{D}_3$. Once the isomorphism $U \cong M(1)^+$ is fixed, the Heisenberg space is one-dimensional, so $\mu = \pm \lambda_0$. Moreover $M(1, \lambda_0) \cong M(1, -\lambda_0)$ as $M(1)^+$ -modules. Consequently

$$U \cdot F \cong Y_1 = M(1, \lambda_0) \in \mathcal{D}_3.$$

Set $W = \langle U, U \cdot F \rangle_{\text{VOA}}$. Proposition 5.4 places W in $\text{Ind}_{\text{CMY}}(\mathcal{D}_3)$. Let

$$B = \widehat{\Psi}_3^{-1}(W).$$

Lemma 3.8 shows that B is generated by ρ_1 , while $W_0 = \mathbb{C}\mathbf{1}$ gives $B^{O_2} = \mathbb{C}\mathbf{1}$. The self-duality ι_ρ therefore gives a surjection

$$\pi_B : \text{Sym}(\rho_1^*) \twoheadrightarrow B, \quad \pi_B(Q_\rho) = c\mathbf{1}.$$

The top vector F is nonisotropic, and

$$F_5 F = -(F, F)\mathbf{1} \neq 0,$$

so the unit component of $\rho_1 \otimes \rho_1 \rightarrow B$ is nonzero. Since the invariant contraction space is one-dimensional, evaluation on $\check{Q}_\rho$ gives $c \neq 0$. Hence $R_c \twoheadrightarrow B$, and Lemma 5.7 makes this map an isomorphism.

Lemma 5.9 gives $B_3^{\text{std}} \cong R_{c_3}$ for the standard extension $V_{\Lambda_3}^+ \supset M(1)^+$, with $c_3 \neq 0$. Choose $a \in \mathbb{C}^\times$ with $a^2 c = c_3$. Scaling the generating copy of ρ_1 by a gives an O_2 -algebra isomorphism $R_{c_3} \cong R_c$. Lemma 5.8 and the extension correspondence now yield $W \cong V_{\Lambda_3}^+$. Its character equals $\text{ch } V$, so equality in every homogeneous degree gives $W = V$. $\square$

6.4. The case $k = 4$: the first-singular map. Assume from now on

$$\text{ch } V = \text{ch } V_{\Lambda_4}^+.$$

Then $\dim P_4(V) = 2$. Let $M_{\text{Vir}}(1, 4)$ be the Verma module with highest-weight vector v_4 , and let s_5 be its first singular vector, unique up to scalar, at level five.

Definition 6.8. For $u \in P_4(V)$ let

$$\phi_u : M_{\text{Vir}}(1, 4) \longrightarrow V, \quad v_4 \longmapsto u$$

be the canonical Virasoro homomorphism. Define the first-singular map

$$S : P_4(V) \longrightarrow V_9, \quad S(u) = \phi_u(s_5).$$

Lemma 6.9 (Double Verma saturation through weight eight). *The space $P_4(V)$ is two-dimensional and nondegenerate. The natural map*

$$\Phi : M_{\text{Vir}}(1, 4) \otimes P_4(V) \longrightarrow V$$

is injective in levels $0, 1, 2, 3, 4$, and for $4 \leq n \leq 8$,

$$V_n = L(1, 0)_n \oplus \Phi(M_{\text{Vir}}(1, 4)_{n-4} \otimes P_4(V)).$$

Proof. The low-weight table gives $V_1 = 0$ and $\dim V_2 = \dim V_3 = 1$, so below weight four there is only the vacuum Virasoro module. Since $\dim V_4 = 4$ and $\dim L(1, 0)_4 = 2$, the orthogonal complement P of $L(1, 0)_4$ in V_4 is two-dimensional and nondegenerate. It is also primary: if $u \in P$, $1 \leq n \leq 4$, and $a \in V_{4-n} = L(1, 0)_{4-n}$, then

$$(L(n)u, a) = (u, L(-n)a) = 0.$$

Nondegeneracy of the form on V_{4-n} gives $L(n)u = 0$, while $L(n)u = 0$ for $n > 4$ by CFT type. This proves $P \subseteq P_4(V)$. Conversely, if $u \in P_4(V)$, then u is orthogonal to $L(1, 0)_4$: every vector in that space is a linear combination of vacuum descendants, and repeated invariance moves their negative Virasoro modes to positive modes annihilating u . Thus $P_4(V) \subseteq P$, while the preceding argument gave $P \subseteq P_4(V)$. Therefore $P = P_4(V)$; in particular, this primary space is two-dimensional and nondegenerate.

Proposition 3.5 says that $M_{\text{Vir}}(1, 4)$ has no singular vector below level five. If Φ had a first nonzero kernel vector at some level $r < 5$, every positive Virasoro mode would send it to a lower-level kernel vector and hence to zero. It would therefore be singular, which is impossible. Thus Φ is injective through level four. Its image is a Virasoro submodule supported in weights at least four. Its intersection with the simple vacuum module $L(1, 0)$ is zero: a nonzero intersection would be all of $L(1, 0)$ and would contain the vacuum. Through weight eight, the character of $V_{\Lambda_4}^+$ is the sum of the vacuum character and two copies of the Verma character $M_{\text{Vir}}(1, 4)$; the first singular relation occurs only at weight nine. This direct sum has the full target dimension in weights four through eight, which proves the formula. $\square$

Lemma 6.10 (The critical-level kernel). *At level five,*

$$\ker \Phi_9 = \mathbb{C}s_5 \otimes \ker S,$$

and hence

$$\dim \text{Im } \Phi_9 = 12 + \text{rank } S.$$

Proof. Let x be a level-five kernel vector. Every positive Virasoro mode sends x to a lower-level kernel vector, which vanishes by Lemma 6.9. Thus x is singular in the direct sum of two copies of $M_{\text{Vir}}(1, 4)$. The level-five singular space of one copy is one-dimensional, spanned by s_5 , so the full singular space is

$$\mathbb{C}s_5 \otimes P_4(V).$$

On this space the map is $s_5 \otimes u \mapsto S(u)$, proving the kernel formula. Since $p(5) = 7$ and $\dim P_4(V) = 2$,

$$\dim \text{Im } \Phi_9 = 14 - \dim \ker S = 14 - (2 - \text{rank } S) = 12 + \text{rank } S.$$

$\square$

Theorem 6.11 (Vanishing of the first-singular map). *In the case $k = 4$,*

$$S = 0.$$

Consequently every nonzero $u \in P_4(V)$ generates an embedded $X_2 = L(1, 4)$.

Proof. Proposition 6.2 gives

$$\dim V_9 = 21, \quad \dim L(1, 0)_9 = 8,$$

and Lemma 6.10 gives

$$\dim \operatorname{Im} \Phi_9 = 12 + \operatorname{rank} S.$$

The image is supported in weights at least four. Its intersection with the simple vacuum module is therefore zero: a nonzero intersection would be the entire vacuum module and would contain **1**. Consequently

$$8 + 12 + \operatorname{rank} S = \dim(L(1, 0)_9 \oplus \operatorname{Im} \Phi_9) \leq 21,$$

so $\operatorname{rank} S \leq 1$.

Assume $\operatorname{rank} S = 1$ and choose u with $s = S(u) \neq 0$. The vector s is a Virasoro primary of weight nine. The dimension equality now gives

$$V_9 = L(1, 0)_9 \oplus \operatorname{Im} \Phi_9.$$

We show that s is orthogonal to both summands. If x is a vacuum descendant, repeated invariance moves all negative Virasoro modes from x to positive modes acting on the primary s , giving $(s, x) = 0$. If

$$y = L(-n_1) \cdots L(-n_r)v$$

is a level-five descendant of a vector $v \in P_4(V)$, then similarly

$$(s, y) = (L(n_r) \cdots L(n_1)s, v) = 0.$$

Thus $s \perp V_9$, contradicting nondegeneracy of the invariant form on the homogeneous space V_9 . Therefore $\operatorname{rank} S = 0$.

For every $u \neq 0$, the first singular vector is therefore killed. Proposition 3.5 says that it generates the maximal proper submodule of $M_{\text{Vir}}(1, 4)$, so ϕ_u factors through a nonzero map $X_2 \rightarrow V$. Simplicity of X_2 makes this map injective. $\square$

6.5. The regular spin-two orbit.

Lemma 6.12 (Orthogonal diagonalization on the regular locus). *Let $A \in \operatorname{Sym}^2(\mathbb{C}^3)$ have three distinct eigenvalues. Then there is $g \in SO_3(\mathbb{C})$ such that gAg^{-1} is diagonal.*

Proof. If $Av = \lambda v$ and $Aw = \mu w$, symmetry gives

$$(\lambda - \mu)(v, w) = (Av, w) - (v, Aw) = 0.$$

Thus distinct eigenspaces are orthogonal. Each eigenline is nondegenerate: an isotropic eigenvector would be orthogonal to its own line and to the other two eigenlines, hence to all of $\mathbb{C}^3$, which is impossible. Normalize one vector on each eigenline to obtain an orthonormal

eigenbasis. Changing the sign of one basis vector if necessary makes the determinant of the change-of-basis matrix equal to one. $\square$

For $A \in E = \text{Sym}_0^2(\mathbb{C}^3)$ define

$$q(A) = \text{tr}(A^2), \quad d(A) = \text{tr}(A^3).$$

Restrict invariant polynomials to diagonal matrices. The nonvanishing of the discriminant defines a nonempty Zariski-open subset of E ; for example, it contains $\text{diag}(1, 0, -1)$. By Lemma 6.12, this regular locus is the G -saturation of the diagonal matrices with distinct eigenvalues and is dense in E . Restriction is therefore injective. Every permutation of the diagonal entries is induced by an element of $SO_3(\mathbb{C})$: if necessary, multiply the permutation matrix by a diagonal sign matrix that centralizes the diagonal matrices and corrects the determinant. The restricted polynomial is therefore symmetric in the eigenvalues x_1, x_2, x_3 , subject to $x_1 + x_2 + x_3 = 0$.

Conversely, every such symmetric polynomial is a polynomial in the elementary functions e_2, e_3 . Newton's identities give

$$e_2 = -\frac{1}{2}q, \quad e_3 = \frac{1}{3}d.$$

Thus every symmetric polynomial on the traceless diagonal is the restriction of a polynomial in q and d , and

$$\mathbb{C}[E]^G = \mathbb{C}[q, d].$$

The discriminant of the traceless characteristic polynomial is

$$\Delta = \frac{1}{2}q^3 - 3d^2.$$

It is nonzero precisely when A has three distinct eigenvalues.

Lemma 6.13 (Regular fibers). *If*

$$\frac{1}{2}c_2^3 - 3c_3^2 \neq 0,$$

then the affine scheme

$$\mathcal{F}_{c_2, c_3} = \text{Spec} \frac{\text{Sym}(E^*)}{(q - c_2, d - c_3)}$$

is smooth and reduced, and

$$\mathcal{F}_{c_2, c_3} \cong G/D_2,$$

where $D_2 \cong (\mathbb{Z}/2)^2$ is the Klein four subgroup of coordinate-axis half-turns.

Proof. For a traceless 3×3 matrix A one has

$$\det(tI_3 - A) = t^3 - \frac{1}{2}q(A)t - \frac{1}{3}d(A).$$

The discriminant hypothesis says that the polynomial

$$t^3 - \frac{1}{2}c_2t - \frac{1}{3}c_3$$

has three distinct roots x_1, x_2, x_3 . Their sum is zero, and Newton's identities give $\sum x_i^2 = c_2$ and $\sum x_i^3 = c_3$. Hence $\text{diag}(x_1, x_2, x_3)$ lies in the fiber, so the fiber is nonempty.

Every matrix in the fiber has these three distinct eigenvalues. Lemma 6.12 diagonalizes it by an element of $SO_3(\mathbb{C})$. Thus all points of the fiber are conjugate. The stabilizer of a diagonal matrix with three distinct eigenvalues is the determinant-one diagonal sign subgroup D_2 , and the closed points form a single orbit G/D_2 .

We must exclude a nonreduced thickening. For $H \in E$,

$$dq_A(H) = 2 \operatorname{tr}(AH), \quad dd_A(H) = 3 \operatorname{tr}(A^2H).$$

The trace pairing restricts nondegenerately to E . Indeed, inside $\operatorname{Sym}_3(\mathbb{C})$ one has $E^\perp = \mathbb{C}I_3$, and therefore $E \cap E^\perp = 0$. Since A has three distinct eigenvalues, in particular $A \neq 0$, so the functional

$$dq_A = 2 \operatorname{tr}(A \cdot)$$

is nonzero on E .

Suppose that the Jacobian of (q, d) had rank less than two at A . Since $dq_A \neq 0$, there would be $\mu \in \mathbb{C}$ such that $dd_A = \mu dq_A$ on E . Hence

$$\operatorname{tr}((3A^2 - 2\mu A)H) = 0 \quad (H \in E).$$

As $E^\perp = \mathbb{C}I_3$, there is $\nu \in \mathbb{C}$ such that

$$3A^2 - 2\mu A = \nu I_3.$$

Thus every eigenvalue of A is a root of the quadratic polynomial $3t^2 - 2\mu t - \nu$, so A could have at most two distinct eigenvalues, a contradiction. Therefore the Jacobian of (q, d) has rank two at every closed point of the fiber. The fiber cut out by $q - c_2$ and $d - c_3$ in the smooth five-dimensional affine space E is consequently smooth of codimension two at every closed point, hence is a smooth three-dimensional affine scheme. In particular, it is reduced.

Fix $A_0 = \text{diag}(x_1, x_2, x_3)$ in the fiber and put $\mathcal{O} = G \cdot A_0$. Standard orbit theory gives an isomorphism

$$G/D_2 \longrightarrow E, \quad gD_2 \longmapsto gA_0g^{-1},$$

onto the locally closed smooth subvariety $\mathcal{O} \subset E$. The diagonalization argument shows that every closed point of $\mathcal{F}_{c_2, c_3}$ lies in $\mathcal{O}$. Since $\mathcal{O}$ is locally closed, the complement

$$|\mathcal{F}_{c_2, c_3}| \setminus \mathcal{O}$$

is constructible. If it were nonempty, it would contain a closed point, because finite-type schemes over $\mathbb{C}$ are Jacobson. This is a contradiction. Hence $|\mathcal{F}_{c_2, c_3}| = \mathcal{O}$, and $\mathcal{O}$ is closed in E .

Give $\mathcal{O}$ its reduced induced scheme structure. The fiber is also reduced, so these are two reduced closed subschemes of E with the same underlying closed set. Their defining ideals are radical and have the same zero locus. The Nullstellensatz therefore gives equality of the ideals, and

$$\mathcal{F}_{c_2, c_3} = \mathcal{O} \cong G/D_2$$

as affine schemes. □

Lemma 6.14 (Invariant parameters on the primary space). *Put $P = P_4(V)$. There is a commutative G -algebra B_P , generated by $E \otimes P$, with the following properties.*

- (i) *For every $u \in P$, the G -subalgebra $B_u \subset B_P$ generated by $E \otimes u$ corresponds to the vertex subalgebra $\langle L(1, 0), u \rangle_{\text{VOA}}$.*
- (ii) *The invariant polynomials q and d take scalar values in B_u . These values are homogeneous forms*

$$q_V(u)\mathbf{1}, \quad d_V(u)\mathbf{1},$$

where $q_V \in \text{Sym}^2(P^)$ and $d_V \in \text{Sym}^3(P^*)$.*

- (iii) *The quadratic form q_V is a nonzero scalar multiple of the invariant bilinear form on P , and in particular is nondegenerate.*

Proof. The maps $M_{\text{Vir}}(1, 4) \rightarrow V$ depend linearly on their highest-weight vectors. Theorem 6.11 and Proposition 3.5 therefore induce a Virasoro homomorphism

$$X_2 \otimes P \longrightarrow V.$$

Its kernel is a submodule of the semisimple module $X_2 \otimes P$. A nonzero submodule would meet the highest-weight space, but the map on that space is the identity $P \rightarrow P_4(V)$. Hence this homomorphism is injective.

Let U_P be the vertex subalgebra generated by $L(1, 0)$ and $X_2 \otimes P$. Theorem 3.7 places U_P in $\text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}})$; let B_P be its commutative G -algebra. By Lemma 3.8, B_P is generated by $E \otimes P$. Moreover $B_P^G = \mathbb{C}\mathbf{1}$, since $(U_P)_0 = \mathbb{C}\mathbf{1}$.

For $u \in P$, the equivariant map

$$j_u : E^* \longrightarrow B_P, \quad \xi \longmapsto \iota_E(\xi) \otimes u,$$

extends to a G -algebra map $\text{Sym}(E^*) \rightarrow B_P$; denote its image by B_u . Thus B_u is the smallest unital G -subalgebra of B_P containing $E \otimes u$.

Set

$$W_u = \langle L(1, 0), u \rangle_{\text{VOA}} \subset U_P.$$

By Theorem 3.7, $W_u \in \text{Ind}_{\text{CMY}}(\mathcal{C}_{\text{sq}})$. Let $B(W_u) \subset B_P$ be the algebra object corresponding to this inclusion. It is a unital G -subalgebra containing $E \otimes u$, so minimality gives $B_u \subseteq B(W_u)$ and hence $\widehat{\Phi}(B_u) \subseteq W_u$.

For the reverse inclusion, put $C_u = \widehat{\Phi}(B_u)$. It is an $L(0)$ -stable submodule of the grading-restricted VOA U_P , hence it is lower bounded and has finite-dimensional homogeneous spaces. Proposition 2.14 therefore shows that C_u is a vertex operator subalgebra of U_P . It contains $L(1, 0)$ and the copy of X_2 with highest-weight vector u , so $W_u \subseteq C_u$. Together with the opposite inclusion obtained by minimality, this gives $C_u = W_u$ and $B_u = B(W_u)$, proving (i).

The images of $q, d \in \text{Sym}(E^*)^G$ under the map defined by j_u lie in $B_P^G = \mathbb{C}\mathbf{1}$. Because j_u depends linearly on u , their scalar values depend homogeneously on u with degrees two and three. This defines q_V and d_V and proves (ii).

To identify q_V , let

$$\text{ev}_E : E \otimes E \longrightarrow \mathbf{1}$$

be the nonzero contraction defined by the trace form. The unit component of multiplication on $(E \otimes P) \otimes (E \otimes P)$ is G -invariant; hence it has the form $\text{ev}_E \otimes \beta_P$ for a bilinear form β_P on P . Commutativity makes β_P symmetric.

The equivalence Φ sends ev_E to a nonzero map

$$X_2 \boxtimes X_2 \longrightarrow X_0.$$

This morphism space is one-dimensional. The canonical contragredient intertwiner spanning it has a nonzero vacuum coefficient on the two highest-weight vectors. Thus there is a scalar $\gamma_E \in \mathbb{C}^\times$ such that the vacuum projection of the product of the primaries $u, v \in P$ is

$$\gamma_E \beta_P(u, v) z^{-8} \mathbf{1}.$$

Lemma 3.4, applied directly in V , identifies the same projection with $(u, v) z^{-8} \mathbf{1}$. Therefore

$$\beta_P(u, v) = \gamma_E^{-1}(u, v).$$

Under the trace self-duality ι_E , the one-dimensional space $(\text{Sym}^2 E^*)^G$ is the line of contraction tensors for the trace pairing. Since $q(A) = \text{tr}(A^2)$ is a nonzero generator of that line, applying the algebra map induced by j_u and projecting to the unit gives

$$q_V(u) = c_E \beta_P(u, u)$$

for a fixed $c_E \in \mathbb{C}^\times$. Thus

$$q_V(u) = c_E \gamma_E^{-1}(u, u).$$

The scalar $c_E \gamma_E^{-1}$ is nonzero, and the invariant form on P is nondegenerate. Hence q_V is nondegenerate, proving (iii). $\square$

Lemma 6.15 (Existence of a regular primary). *There exists $u \in P_4(V)$ such that*

$$\frac{1}{2}q_V(u)^3 - 3d_V(u)^2 \neq 0.$$

Equivalently, the scalar values $c_2(u) = q_V(u)$ and $c_3(u) = d_V(u)$ in B_u have nonzero discriminant.

Proof. If the assertion failed, the discriminant polynomial would vanish at every point of $P_4(V)$. Since $\mathbb{C}$ is infinite, the polynomial identity

$$\frac{1}{2}q_V^3 = 3d_V^2$$

would hold on the two-dimensional space $P_4(V)$. In particular, d_V would be nonzero. By Lemma 6.14, q_V is a nondegenerate binary quadratic form. Over $\mathbb{C}$ it factors as

$$q_V = \ell_1 \ell_2$$

with distinct linear factors. Each factor occurs with odd multiplicity three in q_V^3 , whereas every irreducible factor of the square d_V^2 occurs with even multiplicity. This is impossible. $\square$

Theorem 6.16 (Regular spin-two reconstruction). *For a regular u as in Lemma 6.15,*

$$\langle L(1, 0), u \rangle_{\text{VOA}} \cong V_{A_1}^{D_2}.$$

Here D_2 is the standard Klein four subgroup constructed in Lemma 6.17.

Proof. Use the notation of Lemma 6.14. By construction, B_u is a G -stable unital subalgebra of B_P . Hence

$$\mathbb{C}\mathbf{1} \subseteq B_u^G \subseteq B_P^G = \mathbb{C}\mathbf{1},$$

so $B_u^G = \mathbb{C}\mathbf{1}$. The fixed self-duality $\iota_E : E^* \xrightarrow{\sim} E$ defines a surjection $\text{Sym}(E^*) \twoheadrightarrow B_u$. Since $\mathbb{C}[E]^G = \mathbb{C}[q, d]$, the two invariants take the scalar values $c_2 = q_V(u)$ and $c_3 = d_V(u)$. Regularity gives

$$\frac{1}{2}c_2^3 - 3c_3^2 \neq 0,$$

and hence a surjective G -algebra map

$$\frac{\text{Sym}(E^*)}{(q - c_2, d - c_3)} \twoheadrightarrow B_u.$$

By Lemma 6.13, the source is the affine homogeneous coordinate algebra $\mathcal{O}(G/H_u)$, where H_u is the stabilizer of a regular spin-two vector and is conjugate in G to a Klein four subgroup. This coordinate algebra has no nonzero proper G -stable ideal by Lemma 2.15, so the surjection is an isomorphism. The subgroup H_u is finite, hence reductive, and G/H_u is affine by Lemma 6.13. Corollary 2.18 therefore gives

$$\langle L(1, 0), u \rangle_{\text{VOA}} \cong V_{A_1}^{H_u}.$$

The classification of finite subgroups of $\text{PSL}_2(\mathbb{C})$ shows that all such Klein four subgroups are conjugate. Thus $H_u = gD_2g^{-1}$ for some $g \in G$, where D_2 is the subgroup in Lemma 6.17. Since g acts by a VOA automorphism,

$$V_{A_1}^{H_u} = g(V_{A_1}^{D_2}) \cong V_{A_1}^{D_2}.$$

This proves the stated isomorphism. □

Lemma 6.17 (The D_2 fixed point). *Inside V_{A_1} one has*

$$V_{A_1}^{D_2} = V_{2A_1}^+.$$

Proof. Write $A_1 = \mathbb{Z}\gamma$ with $\langle \gamma, \gamma \rangle = 2$. Define commuting involutions

$$\sigma|_{M(1)} = \text{id}, \quad \sigma(e^{n\gamma}) = (-1)^n e^{n\gamma},$$

and the lattice inversion

$$\theta(\gamma(-m)) = -\gamma(-m), \quad \theta(e^{n\gamma}) = e^{-n\gamma}.$$

On the weight-one basis

$$h = \gamma(-1)\mathbf{1}, \quad x = e^\gamma + e^{-\gamma}, \quad y = e^\gamma - e^{-\gamma},$$

the invariant form makes the three lines pairwise orthogonal and is nonzero on each of them. After replacing y by iy and rescaling the three vectors, they form an orthonormal basis of

the standard three-dimensional $SO_3(\mathbb{C})$ -module. In the original basis the two involutions act as

$$\sigma : (h, x, y) \mapsto (h, -x, -y), \quad \theta : (h, x, y) \mapsto (-h, x, -y).$$

Thus their matrices are

$$\text{diag}(1, -1, -1), \quad \text{diag}(-1, 1, -1),$$

and they generate precisely the determinant-one diagonal sign subgroup, the standard Klein four subgroup D_2 .

The σ -fixed lattice charges are the even charges, so

$$V_{A_1}^{(\sigma)} = V_{2A_1}.$$

The restriction of θ to this fixed-point algebra is exactly lattice inversion for $2A_1$. Therefore

$$V_{A_1}^{D_2} = \left(V_{A_1}^{(\sigma)} \right)^{(\theta)} = V_{2A_1}^+.$$

□

Theorem 6.18 (The case $k = 4$). *If $\text{ch } V = \text{ch } V_{\Lambda_4}^+$, then*

$$V \cong V_{\Lambda_4}^+.$$

Proof. Theorem 6.11 shows that every nonzero line in the two-dimensional space $P_4(V)$ gives an embedded X_2 . Lemma 6.15 provides a regular choice u . By Theorem 6.16 and Lemma 6.17,

$$\langle L(1, 0), u \rangle \cong V_{A_1}^{D_2} = V_{2A_1}^+.$$

The generator of $2A_1$ has squared length eight, so this is exactly $V_{\Lambda_4}^+$. Hence the generated subVOA has the same graded character as V . Since it is a graded subspace of V , equality of homogeneous dimensions implies equality with V . □

Theorem 6.19 (Rank-one lattice-orbifold rigidity). *Let L be any positive-definite even lattice of rank one, and let V satisfy the hypotheses of Theorem 1.1. If*

$$\text{ch } V = \text{ch } V_L^+,$$

then

$$V \cong V_L^+.$$

No Virasoro complete reducibility assumption is required.

Proof. Write $L = \Lambda_k$ with generator squared length $2k$. The cases $k > 4$ are Theorem 5.10; the cases $k = 1, 2, 3, 4$ are Theorems 6.3, 6.4, 6.7, and 6.18, respectively. □

7. THE LATTICE BRANCH AND THE FINAL CLASSIFICATION

7.1. The lattice character branch. We first isolate the only remaining structural branch. The following is the form of the effective-central-charge theorem that we need.

Theorem 7.1 (Dong–Mason). *Let V be strongly rational, and let ℓ be the Lie rank of the reductive Lie algebra V_1 . Then*

$$\ell \leq \tilde{c}(V).$$

If moreover

$$\ell = \tilde{c}(V) = c(V),$$

then V is isomorphic to the lattice vertex operator algebra V_K of a positive-definite even lattice K of rank ℓ .

Proof. Dong–Mason include strong CFT type in their definition of strong rationality. For a simple CFT-type VOA, Li’s criterion [14, Corollary 3.2] identifies self-contragredience with $L(1)V_1 = 0$, so our convention implies theirs. The result is then the combination of Theorems 1–3 of [11]: Theorem 1 gives reductivity of V_1 , Theorem 2 gives the inequality between its Lie rank and the effective central charge, and Theorem 3 identifies the equality case $\ell = \tilde{c} = c$ with a positive-definite even lattice VOA. $\square$

Theorem 7.2 (Rigidity of the rank-one lattice character). *Let V satisfy the hypotheses of Theorem 1.1, and let L be a positive-definite even lattice of rank one. If*

$$\text{ch } V = \text{ch } V_L,$$

then

$$V \cong V_L.$$

Proof. A rank-one lattice VOA contains the rank-one Heisenberg vector, so $(V_L)_1 \neq 0$. Hence character equality implies $V_1 \neq 0$. Since V is strongly rational, V_1 is reductive. Therefore its Lie rank ℓ is at least one. On the other hand, Theorem 7.1 and $\tilde{c}(V) = 1$ give

$$\ell \leq 1.$$

Consequently

$$\ell = 1 = \tilde{c}(V) = c(V).$$

The equality statement in Theorem 7.1 gives

$$V \cong V_K$$

for a positive-definite even lattice K of rank one.

For a positive-definite even lattice M of rank one,

$$\text{ch } V_M = \frac{\Theta_M}{\eta},$$

where $\eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$. Thus

$$\Theta_K = \Theta_L.$$

Write $K = \mathbb{Z}\beta$ and $L = \mathbb{Z}\alpha$ with

$$\langle \beta, \beta \rangle = 2r, \quad \langle \alpha, \alpha \rangle = 2s, \quad r, s \in \mathbb{Z}_{>0}.$$

Then

$$\Theta_K(q) = \sum_{n \in \mathbb{Z}} q^{rn^2}, \quad \Theta_L(q) = \sum_{n \in \mathbb{Z}} q^{sn^2}.$$

The smallest positive exponent appearing in these theta series is respectively r and s , so equality implies $r = s$. Therefore $K \cong L$ as even lattices, and hence $V \cong V_L$. $\square$

7.2. Proof of the classification theorem.

Proof of Theorem 1.1. By Theorem 2.2, the vacuum character of V is one of

$$\text{ch } V_L, \quad \text{ch } V_L^+, \quad \text{ch } V_{A_1}^{A_4}, \quad \text{ch } V_{A_1}^{S_4}, \quad \text{ch } V_{A_1}^{A_5},$$

for a positive-definite even rank-one lattice L .

If $\text{ch } V = \text{ch } V_L$, Theorem 7.2 gives $V \cong V_L$.

If $\text{ch } V = \text{ch } V_L^+$, Theorem 6.19 gives

$$V \cong V_L^+.$$

This includes all lattice norms, including the four collision cases treated in Section 6.

Finally, suppose

$$\text{ch } V = \text{ch } V_{A_1}^H, \quad H \in \{A_4, S_4, A_5\}.$$

The exceptional character-rigidity theorem, Theorem 3.13, gives

$$V \cong V_{A_1}^H.$$

These five alternatives exhaust the character list. At no point in the rigidity argument was complete reducibility of V as an $L(1, 0)$ -module assumed. This proves the theorem. $\square$

Corollary 7.3. *Under the hypotheses of Theorem 1.1, the additional assumption used in several earlier characterization arguments that V be a direct sum of highest-weight $L(1, 0)$ -modules is unnecessary.*

Proof. The character reduction in Theorem 2.2 uses only rationality, C_2 -cofiniteness, self-contragredience, and $c = \tilde{c} = 1$; it does not require Virasoro semisimplicity. The lattice branch is Theorem 7.2. Theorem 6.19 treats the rank-one orbifolds by constructing only the semisimple subcores needed for reconstruction, and Theorem 3.13 treats the three exceptional branches in the same way. Thus no branch uses a global decomposition of V into Virasoro highest-weight modules. $\square$

Remark 7.4 (Strong rationality of the models). *Positive-definite lattice VOAs are simple, regular, non-negatively graded, and equipped with a nonsingular invariant bilinear form. Carnahan–Miyamoto’s finite-solvable-orbifold theorem therefore applies [3, Corollary 5.25]. Lattice inversion is cyclic, and A_4 and S_4 are solvable; their fixed-point algebras remain simple, regular, non-negatively graded, and self-contragredient. The degree-zero space is still $\mathbb{C}1$, so these algebras are of CFT type. Together with the equivalence between regularity and rationality plus C_2 -cofiniteness [1, Theorem 4.5], this proves their strong rationality.*

For the remaining icosahedral model, Xu proves directly that $V_{A_1}^{A_5}$ is strongly rational [21]. Thus every VOA on the classification list is strongly rational.

APPENDIX A. WHY AMBIENT VIRASORO SEMISIMPLICITY CANNOT BE USED AS A FORMAL SHORTCUT

For completeness, we record a simple obstruction that explains why the generated-core arguments above are needed. Even at central charge one, integral grading, diagonalizable $L(0)$ and trivial twist do not force a Virasoro module to be semisimple.

Proposition A.1 (Explicit nonsplit square blocks). *For every $m \geq 0$, fix a singular-vector embedding*

$$\iota_m : M_{\text{Vir}}(1, (m+1)^2) \hookrightarrow M_{\text{Vir}}(1, m^2)$$

and identify its source with its image. Define

$$E_m := M_{\text{Vir}}(1, m^2) / (\iota_m \circ \iota_{m+1}) M_{\text{Vir}}(1, (m+2)^2).$$

Then E_m fits into a nonsplit exact sequence

$$0 \longrightarrow L(1, (m+1)^2) \longrightarrow E_m \longrightarrow L(1, m^2) \longrightarrow 0.$$

Moreover, $L(0)$ acts semisimply on E_m , all conformal weights are integers, and the categorical twist is the identity.

Proof. Proposition 3.5 supplies the embeddings ι_m , unique up to nonzero scalar. Once a normalization is fixed, the Feigin–Fuchs chain is

$$M_{\text{Vir}}(1, m^2) \supset \iota_m M_{\text{Vir}}(1, (m+1)^2) \supset (\iota_m \circ \iota_{m+1}) M_{\text{Vir}}(1, (m+2)^2) \supset \cdots.$$

The quotient of the first two nested submodules is the irreducible module

$$\iota_m M_{\text{Vir}}(1, (m+1)^2) / (\iota_m \circ \iota_{m+1}) M_{\text{Vir}}(1, (m+2)^2) \cong L(1, (m+1)^2),$$

and the maximal proper submodule of $M_{\text{Vir}}(1, m^2)$ is $\iota_m M_{\text{Vir}}(1, (m+1)^2)$. Passing to the indicated quotient therefore gives an exact sequence

$$0 \rightarrow L(1, (m+1)^2) \rightarrow E_m \rightarrow L(1, m^2) \rightarrow 0.$$

If the sequence split, E_m would contain a highest-weight vector v' of weight m^2 whose generated submodule is isomorphic to $L(1, m^2)$. The image of v' in the quotient $L(1, m^2)$ is nonzero, so after rescaling it differs from the image of the original Verma highest-weight vector by a vector from the submodule of lowest weight $(m+1)^2$. Since that submodule has no vector of weight m^2 , the weight- m^2 space of E_m is one-dimensional and v' must be the image of the original Verma highest-weight vector. But its first singular descendant, at weight $(m+1)^2$, survives in E_m by construction. Hence the cyclic module generated by v' is not the irreducible quotient $L(1, m^2)$, a contradiction. Thus the sequence is nonsplit.

The same sequence is nonsplit in the category of $L(1, 0)$ -modules: any splitting there would, after restriction to the Virasoro action, give a Virasoro-module splitting.

The module E_m is a highest-weight Virasoro module of central charge one: it is a homogeneous quotient of $M_{\text{Vir}}(1, m^2)$ generated by the image of the Verma highest-weight vector. It is lower bounded, its $L(0)$ -eigenspaces are finite dimensional, and for every $w \in E_m$ one has $L(n)w = 0$ for all sufficiently large n . Thus E_m is a restricted Virasoro module. The standard Virasoro–VOA correspondence [4, Section 2.2] therefore makes it a module for the universal Virasoro vertex operator algebra $M(1, 0)$. By [4, Corollary 2.2.5(4)], $M(c, 0)$ is simple unless

$$c = c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}$$

for coprime integers $p, q \geq 2$. If $c_{p,q} = 1$, then $p = q$; coprimality would force $p = q = 1$, contrary to $p, q \geq 2$. Hence central charge one is not exceptional and

$$M(1, 0) = L(1, 0).$$

Consequently E_m is an $L(1, 0)$ -module.

The equalities

$$m^2 = h_{2m+1,1}(1), \quad (m+1)^2 = h_{2m+3,1}(1)$$

show that both lowest weights belong to the degenerate set H_1 of [4, Section 2.2]. The exact sequence above shows that E_m has finite length with composition factors $L(1, m^2)$ and $L(1, (m+1)^2)$. Hence $E_m \in \mathcal{O}_1^{\text{fin}}$ by definition. The equality

$$\mathcal{C}_1 = \mathcal{O}_1^{\text{fin}}$$

of [4, Theorem 3.1.4] now gives $E_m \in \mathcal{O}_1$. In particular, its categorical twist is defined.

A Verma module is an algebraic direct sum of its $L(0)$ -eigenspaces, and the same is true of every homogeneous quotient. Since $m^2 \in \mathbb{Z}$, the spectrum of E_m is contained in $m^2 + \mathbb{Z}_{\geq 0} \subset \mathbb{Z}$. Therefore $L(0)$ is semisimple and, for the standard twist $\theta_W = e^{2\pi i L(0)}$ in the vertex tensor category,

$$e^{2\pi i L(0)} = 1$$

on E_m . This proves the final assertions. $\square$

Remark A.2. *Proposition A.1 shows that neither semisimplicity of $L(0)$, integrality of conformal weights, triviality of the twist, nor equality of graded characters with a sum of irreducible characters is sufficient to deduce complete reducibility as a Virasoro module. The proofs in Sections 3–6 instead extract particular simple submodules and prove that the vertex subalgebras they generate remain in explicitly semisimple tensor subcategories.*

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REFERENCES

- [1] T. Abe, G. Buhl and C. Dong, *Rationality, regularity, and C_2 -cofiniteness*, Trans. Amer. Math. Soc. **356** (2004), 3391–3402.
- [2] D. Adamović, X. Lin and J. Yang, *Tensor category of $\mathbb{Z}_2$ -orbifold of Heisenberg vertex operator algebra and its applications*, arXiv:2604.12120v1 (2026).

- [3] S. Carnahan and M. Miyamoto, *Regularity of fixed-point vertex operator subalgebras*, arXiv:1603.05645 (2016), revised 2018.
- [4] T. Creutzig, C. Jiang, F. Orosz Hunziker, D. Ridout and J. Yang, *Tensor categories arising from the Virasoro algebra*, Adv. Math. **380** (2021), 107601.
- [5] T. Creutzig, S. Kanade and R. McRae, *Tensor categories for vertex operator superalgebra extensions*, Mem. Amer. Math. Soc. **295** (2024), no. 1472.
- [6] T. Creutzig, R. McRae and J. Yang, *Direct limit completions of vertex tensor categories*, Commun. Contemp. Math. **24** (2022), no. 2, 2150033.
- [7] C. Dong and R. L. Griess Jr., *Rank one lattice type vertex operator algebras and their automorphism groups*, J. Algebra **208** (1998), 262–275.
- [8] C. Dong and C. Jiang, *A characterization of the vertex operator algebra $V_{L_2}^{A_4}$* , in: Conformal Field Theory, Automorphic Forms and Related Topics, Contrib. Math. Comput. Sci. **8**, Springer, Heidelberg, 2014, 55–74.
- [9] C. Dong and C. Jiang, *A characterization of vertex operator algebras $V_{\mathbb{Z}\alpha}^+$: I*, J. Reine Angew. Math. **709** (2015), 51–79.
- [10] C. Dong and C. Jiang, *A characterization of the rational vertex operator algebra $V_{\mathbb{Z}\alpha}^+$: II*, Adv. Math. **247** (2013), 41–70.
- [11] C. Dong and G. Mason, *Rational vertex operator algebras and the effective central charge*, Int. Math. Res. Not. 2004, no. 56, 2989–3008.
- [12] Y.-Z. Huang, A. Kirillov Jr. and J. Lepowsky, *Braided tensor categories and extensions of vertex operator algebras*, Comm. Math. Phys. **337** (2015), 1143–1159.
- [13] E. Kiritsis, *Proof of the completeness of the classification of rational conformal theories with $c = 1$* , Phys. Lett. B **217** (1989), 427–430.
- [14] H. Li, *Symmetric invariant bilinear forms on vertex operator algebras*, J. Pure Appl. Algebra **96** (1994), 279–297.
- [15] V. Kac and A. Raina, *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras*, World Scientific, Singapore, 1987.
- [16] A. Milas, *Fusion rings for degenerate minimal models*, J. Algebra **254** (2002), no. 2, 300–335.
- [17] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer, Berlin, 1994.
- [18] Y. Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J. **16** (1960), 205–218.
- [19] R. McRae, *On the tensor structure of modules for compact orbifold vertex operator algebras*, Math. Z. **296** (2020), 409–452.
- [20] R. McRae and J. Yang, *An $\mathfrak{sl}_2$ -type tensor category for the Virasoro algebra at central charge 25 and applications*, Math. Z. **303** (2023), Article 32.
- [21] S. Xu, *C_2 -cofiniteness and rationality of the icosahedral orbifold $V_{L_2}^{A_5}$* , arXiv:2608.13872v1 (2026).
- [22] W. Zhang and C. Dong, *W -algebra $W(2, 2)$ and the vertex operator algebra $L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$* , Comm. Math. Phys. **285** (2009), 991–1004.

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