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Bimodules and universal enveloping algebras associated to SVOAs

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ABSTRACT

For a vertex operator superalgebra V and $n, m \in (1/2)\mathbb{Z}_+$, let $A_n(V) := V/O_n(V)$ denote the associative algebra, and $A_{n,m}(V) := V/O_{n,m}(V)$ denote the $A_n(V) - A_m(V)$-bimodule, as constructed by W. Jiang and C. Jiang [10], where $O_n(V)$ and $O_{n,m}(V)$ are specific subspaces of V. We introduce a novel representation-theoretic method for constructing subspaces $\mathcal{O}_{n,m}(V)$ of V, similar to our previous work [8], and set $\mathcal{O}_n(V) = \mathcal{O}_{n,n}(V)$. We demonstrate that $\mathcal{O}_{n,m}(V) = O_{n,m}(V)$ and $\mathcal{O}_n(V) = O_n(V)$ through a method that is notably simpler and more straightforward compared to the approach detailed in [6] (also see [8]). Moreover, we offer a simpler definition for the bimodules $A_{n,m}(V)$, contributing towards the resolution of a conjecture proposed by Dong and Jiang [2] regarding superalgebras. Additionally, we demonstrate that the $A_n(V) - A_m(V)$-bimodule $A_{n,m}(V)$ is a quotient of $U(V)_{n-m}$, where $U(V)$ denotes the universal enveloping algebra of V, employing a method distinct from [6] (see also [8]), which is unified and simpler.

1. Introduction

Let V be a vertex operator superalgebra. To study the representation theory of V, the associative algebra $A(V)$ is constructed by Kac and Wang [11] (see also [13]). A bijective correspondence was established between the (isomorphism classes of) irreducible admissible V-modules and irreducible $A(V)$-modules. For an admissible V-module $M = \bigoplus_{n\in(1/2)\mathbb{Z}_+} M(n)$, $M(0)$ becomes an $A(V)$-module. To study more $M(n)$, W. Jiang and C. Jiang [10] (see also [4,3]) constructed associative algebras $A_n(V)$ for each $n \in (1/2)\mathbb{Z}_+$, where $A_0(V) = A(V)$, such that for $0 \le k \in (1/2)\mathbb{Z} \le n$, $M(k)$ is an $A_n(V)$-module. Let U be an $A_m(V)$-module which cannot factor through $A_{m-1/2}(V)$, in [10], a Verma-type admissible V-module $\bar{M}(U)$ is constructed such that $\bar{M}(U)(m) = U$ and $\bar{M}(U)(0) \neq 0$. However, we do not know the explicit form of $\bar{M}(U)(k)$ for $k \neq m$. To overcome this issue, for $n, m \in (1/2)\mathbb{Z}_+$, in [10] (see also [5,2]), they constructed the $A_n(V) - A_m(V)$-bimodule $A_{n,m}(V)$ such that $\bigoplus_{n\in(1/2)\mathbb{Z}_+} A_{n,m}(V) \otimes_{A_m(V)} U$ is isomorphic to $\bar{M}(U)$.

The bimodule $A_{n,m}(V)$ is defined as the quotient of V by $O_{n,m}(V)$, and the associative algebra $A_n(V)$ is defined as the quotient of V by $O_n(V)$, where $O_n(V)$ and $O_{n,m}(V)$ are spans of certain specifically defined

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products within V. To study the representation-theoretic significance of $O_n(V)$ and $O_{n,m}(V)$, we define a subspace $\mathcal{O}_{n,m}(V)$ of V in an extrinsic manner, utilizing representation theory:

$$\mathcal{O}_{n,m}(V) = \left\{u \in V \mid o_{m-n}(u)|_{\Omega_m(M)} = 0 \text{ for all weak } V\text{-modules } M\right\}.$$

Set $\mathcal{O}_n(V) = \mathcal{O}_{n,n}(V)$. For the non-super case, Han [6] proved that $\mathcal{O}_{n,m}(V) = O_{n,m}(V)$ and $\mathcal{O}_n(V) = O_n(V)$. For the twisted case of vertex operator algebra, see our previous work [8]. For any vertex operator algebra V, finite automorphism g of V of order T and $m, n \in (1/T)\mathbb{Z}_+$, we construct a family of associative algebras $\mathcal{A}_{g,n}(V) := V/\mathcal{O}_{g,n}(V)$ and $\mathcal{A}_{g,n}(V) - \mathcal{A}_{g,m}(V)$-bimodules $\mathcal{A}_{g,n,m}(V) := V/\mathcal{O}_{g,n,m}(V)$ from the point of view of representation theory, where $\mathcal{O}_{g,n}(V)$ and $\mathcal{O}_{g,n,m}(V)$ are defined similarly to the non-twisted case. We prove that the algebra $\mathcal{A}_{g,n}(V)$ is identical to the algebra $A_{g,n}(V) := V/O_{g,n}(V)$ constructed by Dong, Li and Mason [3], and that the bimodule $\mathcal{A}_{g,n,m}(V)$ is identical to $A_{g,n,m}(V) := V/O_{g,n,m}(V)$ which was constructed by Dong and Jiang [1]. Returning to the case of vertex operator superalgebras, in this paper, we show that $\mathcal{O}_{n,m}(V) = O_{n,m}(V)$ and $\mathcal{O}_n(V) = O_n(V)$ through a method that is notably simpler and more straightforward compared to the approach detailed in [6] (also see [8]), thus providing a unified definition for $A_n(V)$ and $A_{n,m}(V)$ (see Theorem 3.2).

Another important associative algebra related to a vertex operator superalgebra V is its universal enveloping algebra $U(V)$. This algebra is crucial because any weak V-module M can be naturally regarded as a $U(V)$-module, and the structure of M as a weak V-module is fully determined by its $U(V)$-module structure. When V is a vertex operator algebra, Frenkel and Zhu [5] noted that the Zhu algebra $A(V)$ is isomorphic to a quotient of $U(V)_0$. It has been established in [9] (see also [7,5]) that $A_n(V)$ is a quotient algebra of $U(V)_0$ for any $n \in \mathbb{Z}_+$. Additionally, Han [6] demonstrated that the $A_n(V) - A_m(V)$-bimodule $A_{n,m}(V)$ is a quotient of $U(V)_{n-m}$ for any $n, m \in \mathbb{Z}_+$. In our previous work [8], we generalized the above approach to the twisted case. We also prove that the $A_{g,n}(V) - A_{g,m}(V)$-bimodule $A_{g,n,m}(V)$ is isomorphic to $U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}$, where $U(V[g])_k$ is the subspace of degree k of the $(1/T)\mathbb{Z}$-graded universal enveloping algebra $U(V[g])$ of V with respect to g and $U(V[g])_k^l$ is some subspace of $U(V[g])_k$. Whether in the untwisted case or the twisted case, their strategy is to first prove the algebra isomorphism [9,7] and then apply the universal property of Verma-type admissible V-modules to prove the bimodule isomorphism [6,8]. Returning to the case of vertex operator superalgebras, in this paper, we demonstrate that the $A_n(V) - A_m(V)$-bimodule $A_{n,m}(V)$ is a quotient of $U(V)_{n-m}$ employing a method distinct from [6] (see also [8]), which is unified and simpler (see Theorem 6.4).

In our previous work [8], we show that all these bimodules $A_{g,n,m}(V)$ associated to the vertex operator algebra V can be defined in a simpler way. In this paper, we will do similar things for the bimodules $A_{n,m}(V) = V/O_{n,m}(V)$ associated to the vertex operator superalgebra V. For technical reasons, $O_{n,m}(V)$ is defined as the sum of three subspaces $O'_{n,m}(V)$, $O''_{n,m}(V)$, and $O'''_{n,m}(V)$. However, it has been conjectured that $O_{n,m}(V) = O'_{n,m}(V)$ (see [2]). We advance toward this conjecture by proving that $O'''_{n,m}(V)$ is superfluous and that $O'_{n,m}(V)$ can be replaced by its subspace $V^{\bar{s}} + L_{n,m}(V)$, where $\overline{\hat{m} - \hat{n}} \neq \bar{s}$. Thus,

$$O_{n,m}(V) = V^{\bar{s}} + L_{n,m}(V) + O''_{n,m}(V) \quad \text{(see Theorem 4.7)},$$

where $\overline{\hat{m} - \hat{n}} \neq \bar{s}$. This refinement simplifies the definition of the bimodules $A_{n,m}(V)$.

The organization of this paper is as follows. In Section 2, we review the definitions of vertex operator superalgebras, weak modules, and admissible modules. In Section 3, we define a subspace $\mathcal{O}_{n,m}(V)$ of V from a representation-theoretic perspective and set $\mathcal{O}_n(V) = \mathcal{O}_{n,n}(V)$. We demonstrate using a simpler approach than [6] (see also [8]) that $\mathcal{O}_{n,m}(V) = O_{n,m}(V)$ and $\mathcal{O}_n(V) = O_n(V)$. In Section 4, we provide a simplified definition of the bimodules $A_{n,m}(V)$, making progress toward the conjecture of Dong and Jiang [2]. In Section 5, we review the definition of the universal enveloping algebra $U(V)$ for vertex operator superalgebras V. In Section 6, we show that the $A_n(V) - A_m(V)$-bimodule $A_{n,m}(V)$ is a quotient of $U(V)_{n-m}$

for any $n, m \in (1/2)\mathbb{Z}_+$, employing a method distinct from that used in [6] (see also [8]), which is unified and simpler.

2. Basics

We recall definitions of the vertex operator superalgebras, weak modules and admissible modules in this section. For $k \in \mathbb{Z}$, let $\bar{k}$ denote the image of k in $\mathbb{Z}/2\mathbb{Z}$.

Definition 2.1. A *vertex operator superalgebra* is a 4-tuple $(V, Y, \mathbf{1}, \omega)$, where $V = \bigoplus_{n\in(1/2)\mathbb{Z}} V_n = V^{\bar{0}} \bigoplus V^{\bar{1}}$ is a $(1/2)\mathbb{Z}$-graded vector space with $\dim V_n < \infty$ for all n and $V_n = 0$ for $n \ll 0$, where $V^{\bar{0}} = \bigoplus_{n\in\mathbb{Z}} V_n$ and $V^{\bar{1}} = \bigoplus_{n\in(1/2)+\mathbb{Z}} V_n$. $\mathbf{1} \in V_0$, $\omega \in V_2$ and Y is a linear map from V to $\operatorname{End} V\left[\left[z, z^{-1}\right]\right]$ sending $u \in V$ to $Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$ satisfying the following axioms:

(1) $Y(\mathbf{1}, z) = \mathrm{id}_V$ and $u_n\mathbf{1} = \delta_{n,-1}u$ for any $n \geq -1$ and $u \in V$;
(2) $u_n v \in V^{\overline{i+j}}$ for any $u \in V^{\bar{i}}, v \in V^{\bar{j}}$ and $n \in \mathbb{Z}$; for any $u, v \in V$, $u_n v = 0$ for $n \gg 0$;
(3) the Virasoro algebra relations hold: $[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}c_V$ for $m, n \in \mathbb{Z}$, where $c_V \in \mathbb{C}$ and $L(m) = \omega_{m+1}$ for $m \in \mathbb{Z}$; $L(0)|_{V_m} = m\,\mathrm{id}_{V_m}$ for $m \in (1/2)\mathbb{Z}$ and $Y(L(-1)u, z) = \frac{d}{dz}Y(u, z)$ for $u \in V$;
(4) for any $u, v \in V, m, l, n \in \mathbb{Z}$, the *Jacobi identity* holds:

$$\sum_{i\geq 0}(-1)^i\binom{l}{i}\left(u_{m+l-i}v_{n+i} - (-1)^{\tilde{u}\tilde{v}}(-1)^l v_{n+l-i}u_{m+i}\right) = \sum_{i\geq 0}\binom{m}{i}\left(u_{l+i}v\right)_{m+n-i},$$

where $\tilde{x} = 0$ for $x \in V^{\bar{0}}$ and $\tilde{x} = 1$ for $x \in V^{\bar{1}}$. Whenever $\tilde{x}$ appears, we always assume that $x \in V^{\bar{0}}$ or $V^{\bar{1}}$.

For any $n \in (1/2)\mathbb{Z}$, elements in V_n are said to be *homogeneous*, and if $u \in V_n$, we define $\operatorname{wt} u = n$. As a convention, whenever $\operatorname{wt} u$ appears, we always assume that u is homogeneous.

Definition 2.2. A *weak V-module* is a vector space M equipped with a linear map from V to $\operatorname{End} M\left[\left[z, z^{-1}\right]\right]$ sending $u \in V$ to $Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$ satisfying the following axioms:

(1) $Y_M(\mathbf{1}, z) = \mathrm{id}_M$;
(2) for any $u \in V, w \in M$, $u_n w = 0$ for $n \gg 0$;
(3) for any $u, v \in V, m, l, n \in \mathbb{Z}$, the following *Jacobi identity* holds on M:

$$\sum_{i\geq 0}(-1)^i\binom{l}{i}\left(u_{m+l-i}v_{n+i} - (-1)^{\tilde{u}\tilde{v}}(-1)^l v_{n+l-i}u_{m+i}\right) = \sum_{i\geq 0}\binom{m}{i}\left(u_{l+i}v\right)_{m+n-i}. \tag{2.1}$$

Definition 2.3. An *admissible V-module* M is a weak V-module that carries a $(1/2)\mathbb{Z}_+$-grading $M = \bigoplus_{n\in(1/2)\mathbb{Z}_+} M(n)$ with $u_m M(n) \subseteq M(\operatorname{wt} u + n - m - 1)$ for any $u \in V, m \in \mathbb{Z}$ and $n \in (1/2)\mathbb{Z}_+$.

3. $A_n(V) - A_m(V)$-Bimodules $A_{n,m}(V)$

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator superalgebra. For any weak V-module M and $n \in (1/2)\mathbb{Z}$, we define a linear map $o_n(\cdot) : V \to \operatorname{End} M$ by $o_n(v) = v_{\operatorname{wt} v-1+n}$, and set $o(\cdot) = o_0(\cdot)$. Note that $o_n(v) = 0$ if $\operatorname{wt} v - 1 + n \notin \mathbb{Z}$. For $n, m \in (1/2)\mathbb{Z}_+$, define

$$\Omega_n(M) = \{w \in M \mid o_{n+i}(v)w = 0 \text{ for all } v \in V \text{ and } 0 < i \in (1/2)\mathbb{Z}\}.$$
$$\mathcal{O}_{n,m}(V) = \left\{u \in V \mid o_{m-n}(u)|_{\Omega_m(M)} = 0 \text{ for all weak } V\text{-modules } M\right\}.$$

And set $\mathcal{O}_n(V) = \mathcal{O}_{n,n}(V)$.

For any $n \in (1/2)\mathbb{Z}$, there exists a unique $\hat{n} \in \{0,1\}$ such that $n = \lfloor n \rfloor + \frac{\hat{n}}{2}$, where $\lfloor \cdot \rfloor$ denotes the floor function. This decomposition is utilized whenever we refer to $n \in (1/2)\mathbb{Z}$. For $r, i \in \{0,1\}$, define $\delta_i(r) = 1$ if $i \geq r$ and $\delta_i(r) = 0$ if $i < r$ in this paper. As a convention, set $\delta_i(2) = 0$ for $i = 0,1$.

For $u \in V^{\bar{r}}, v \in V$ and $n, m, p \in (1/2)\mathbb{Z}$, define the product $*^n_{m,p}$ on V as follows:

$$u *^n_{m,p} v = \sum_{j=0}^{\lfloor p \rfloor} (-1)^j \binom{\lfloor m \rfloor + \lfloor n \rfloor - \lfloor p \rfloor + \varepsilon + j}{j} \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, u + \lfloor m \rfloor + \delta_{\hat{m}}(r) - 1 + r/2}}{z^{\lfloor m \rfloor + \lfloor n \rfloor - \lfloor p \rfloor + \varepsilon + j + 1}} Y(u,z)v,$$

if $\overline{\hat{p} - \hat{n}} = \bar{r}$ and $n, m, p \geq 0$, where $\varepsilon = -1 + \delta_{\hat{m}}(r) + \delta_{\hat{n}}(2-r)$; and $u *^n_{m,p} v = 0$ otherwise. Set $*^n_m = *^n_{m,m}$, $\bar{*}^n_m = *^n_{m,n}$ and $*_n = *^n_n = \bar{*}^n_n$. It is easy to see that

$$\mathbf{1} \bar{*}^n_m v = v \text{ for } v \in V. \tag{3.1}$$

Let $m, n \in (1/2)\mathbb{Z}_+$, define $O'_{n,m}(V) = V^{\bar{r}} + \mathrm{span}\{u \circ^n_m v \mid u, v \in V\} + L_{n,m}(V)$, where $\overline{\hat{m} - \hat{n}} \neq \bar{r}$, $L_{n,m}(V) = \mathrm{span}\{(L(-1) + L(0) + m - n)u \mid u \in V^{\bar{s}} \text{ such that } \overline{\hat{m} - \hat{n}} = \bar{s}\}$ and for $u, v \in V$,

$$u \circ^n_m v = \begin{cases} \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}(u) + \lfloor m \rfloor}}{z^{\lfloor m \rfloor + \lfloor n \rfloor + 2}} Y(u,z)v, & \text{if } u \in V^{\bar{0}}, \\ \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}(u) + \lfloor m \rfloor + \delta_{\hat{m}}(1) - 1/2}}{z^{\lfloor m \rfloor + \lfloor n \rfloor + \delta_{\hat{m}}(1) + \delta_{\hat{n}}(1) + 1}} Y(u,z)v, & \text{if } u \in V^{\bar{1}}. \end{cases}$$

Set $O_n(V) = O'_{n,n}(V)$ and $A_n(V) = V/O_n(V)$. For any $u, a, b, c \in V$ and any $p_1, p_2, p_3 \in (1/2)\mathbb{Z}_+$, we define $O''_{n,m}(V)$ as the linear span of

$$u *^n_{m,p_3} \left(\left(a *^{p_3}_{p_1,p_2} b\right) *^{p_3}_{m,p_1} c - a *^{p_3}_{m,p_2} \left(b *^{p_2}_{m,p_1} c\right) \right).$$

Define $O'''_{n,m}(V) = \sum_{p_1,p_2 \in (1/2)\mathbb{Z}_+} \left(V *^n_{p_1,p_2} O'_{p_2,p_1}(V)\right) *^n_{m,p_1} V$, $O_{n,m}(V) = O'_{n,m}(V) + O''_{n,m}(V) + O'''_{n,m}(V)$ and $A_{n,m}(V) = V/O_{n,m}(V)$. Take $u = \mathbf{1}$ and $p_3 = n$, by (3.1), we obtain

$$\left(a *^n_{p_1,p_2} b\right) *^n_{m,p_1} c - a *^n_{m,p_2} \left(b *^{p_2}_{m,p_1} c\right) \in O''_{n,m}(V). \tag{3.2}$$

The subsequent theorem is derived from [10, Theorem 3.2, Theorem 3.5, Theorem 3.7 and Theorem 4.7].

Theorem 3.1. (1) *The product $*_n$ induces an associative algebra structure on $A_n(V)$ with the identity element given by $\mathbf{1} + O_n(V)$.*

(2) *For a weak V-module M, $\Omega_n(M)$ is an $A_n(V)$-module induced by the map $a \mapsto o(a)$ for $a \in V^{\bar{0}}$. If $M = \bigoplus_{k \in (1/2)\mathbb{Z}_+} M(k)$ is an admissible V-module, then $\bigoplus_{0 \leq k \in (1/2)\mathbb{Z} \leq n} M(k) \subseteq \Omega_n(M)$, and $M(k)$ is an $A_n(V)$-module for $0 \leq k \in (1/2)\mathbb{Z} \leq n$.*

(3) *For any $A_n(V)$-module U, there exists an admissible V-module $\bar{M}(U)$ such that $\bar{M}(U)(n) = U$.*

(4) *$A_{n,m}(V)$ is an $A_n(V) - A_m(V)$-bimodule for $n, m \in (1/2)\mathbb{Z}_+$, where the left and right actions of $A_n(V)$ and $A_m(V)$ are induced by $\bar{*}^n_m$ and $*^n_m$, respectively.*

Let U be an $A_m(V)$-module. Define $M(U) = \bigoplus_{n \in (1/2)\mathbb{Z}_+} A_{n,m}(V) \otimes_{A_m(V)} U$. Then $M(U)$ is $(1/2)\mathbb{Z}_+$-graded with $M(U)(n) = A_{n,m}(V) \otimes_{A_m(V)} U$ for the convention that $M(U)(i) = 0$ if $i < 0$. For $u, v \in V, w \in U$, $p \in \mathbb{Z}$, and $n \in (1/2)\mathbb{Z}_+$, set $d = n + \mathrm{wt}\, u - p - 1$, define a linear map u_p on $M(U)(n)$ mapping to $M(U)(d)$ by $u_p\left((v + O_{n,m}(V)) \otimes w\right) = \left(u *^d_{m,n} v + O_{d,m}(V)\right) \otimes w$, if $d \geq 0$; and $u_p\left((v + O_{n,m}(V)) \otimes w\right) = 0$

otherwise. Then we form a generating function $Y_{M(U)}(u,z) = \sum_{p\in\mathbb{Z}} u_p z^{-p-1}$. And $M(U)$ is an admissible V-module by [10, Theorem 6.13].

Theorem 3.2. *For any $n, m \in (1/2)\mathbb{Z}_+$, $O_n(V) = \mathcal{O}_n(V)$ and $O_{n,m}(V) = \mathcal{O}_{n,m}(V)$.*

Proof. Consider the admissible V-module $M(A_m(V)) = \bigoplus_{k\in(1/2)\mathbb{Z}_+} A_{k,m}(V)$. By Theorem 3.1 (2), we have

$$A_{m,m}(V) = M(A_m(V))(m) \subseteq \Omega_m(M(A_m(V))).$$

For any $u \in \mathcal{O}_{n,m}(V)$, by the definition of $\mathcal{O}_{n,m}(V)$ and Theorem 3.1 (4), we have

$$0 = o_{m-n}(u)\,(\mathbf{1} + O_{m,m}(V)) = u *_m^n \mathbf{1} + O_{n,m}(V) = u + O_{n,m}(V),$$

which implies $\mathcal{O}_{n,m}(V) \subseteq O_{n,m}(V)$. By [10, Corollary 6.3], $O_{n,m}(V) \subseteq \mathcal{O}_{n,m}(V)$. Thus $\mathcal{O}_{n,m}(V) = O_{n,m}(V)$. Consider admissible V-module $\bar{M}(A_n(V))$ from Theorem 3.1 (3), so $\bar{M}(A_n(V))(n) = A_n(V) \subseteq \Omega_n(\bar{M}(A_n(V)))$ by Theorem 3.1 (2). For any $u \in \mathcal{O}_n(V)$, we have

$$0 = o(u)(\mathbf{1} + O_n(V)) = u *_n \mathbf{1} + O_n(V) = u + O_n(V),$$

which implies $\mathcal{O}_n(V) \subseteq O_n(V)$, then $O_n(V) = \mathcal{O}_n(V)$. □

According to [6, Remark 3.4], it is hard to give a direct proof of $O_n(V) = \mathcal{O}_n(V)$ and $O_{n,m}(V) = \mathcal{O}_{n,m}(V)$. However, we provide a simple and direct proof in Theorem 3.2.

4. Refining bimodules

In this section, we will provide a refined definition of the $A_n(V)$–$A_m(V)$-bimodule $A_{n,m}(V)$ using method in our previous work [8, Section 6] (see also [6]).

Notation 4.1. For the purposes of this discussion, we adopt the following conventions:

(1) For $m \in (1/2)\mathbb{Z}_+$ and $i \in \mathbb{Z}$, define $\binom{m}{i}$ to be 1 if $i = 0$, and 0 if $i < 0$.

(2) For $k, l \in (1/2)\mathbb{Z}$, we define the sum $\sum_{i=k}^{l} a_i$ as $\sum_{i\in\mathbb{Z}_{k,l}} a_i$, where $\mathbb{Z}_{k,l} = \mathbb{Z} \cap [l,k]$, if $l \le k$; and $\mathbb{Z}_{k,l} = \mathbb{Z} \cap [k,l]$ otherwise.

(3) For $n \in (1/2)\mathbb{Z}_+$, $a \in V^{\bar{r}}$, and $b \in V$, set $q = -1 + \lfloor n \rfloor + \delta_{\hat{n}}(r) + r/2$, define

$$f_i(a,b) = \frac{(1+z)^{\mathrm{wt}\,a+q}}{z^i} Y(a,z)b \quad \text{for } i \in \mathbb{Z}.$$

In the subsequent lemmas, Notation 4.1 (2) will be utilized. Setting $T = 2$ in [8, Lemma 6.2], then we have:

Lemma 4.2. *Let $n \in (1/2)\mathbb{Z}$ and $l \in \mathbb{Z}$. Then, the following identity holds:*

$$\sum_{j=0}^{n+1+l} (-1)^j \binom{l}{j} \sum_{i=0}^{n+1+l-j} (-1)^i \binom{-l+i+j-1}{i} \frac{1}{z^{i+j}} = 1.$$

Lemma 4.3. *Let $n, k \in (1/2)\mathbb{Z}_+$, $a \in V^{\bar{r}}$, $b \in V$ and $l, j \in \mathbb{Z}$, set $q = -1 + \lfloor n \rfloor + \delta_{\hat{n}}(r) + r/2$, then the following identity holds:*

$$a *^k_{n,k+1+q+l-j} b = \sum_{i=0}^{k+1+q+l-j} (-1)^i \binom{-l+i+j-1}{i} \mathrm{Res}_z\, f_{i+j-l}(a,b).$$

Proof. Observe that $\lfloor k+1+q+l-j \rfloor = \lfloor n \rfloor + \lfloor k+r/2 \rfloor + \delta_{\hat{n}}(r) + l - j$ and $\lfloor k \rfloor - \lfloor k+r/2 \rfloor + \delta_{\hat{k}}(2-r) = 0$. The lemma follows from the definition of the product $*^n_{m,p}$ and Notation 4.1 (2)-(3). □

Let $m \in (1/2)\mathbb{Z}_+$, set $M^{(m)} = \bigoplus_{n \in (1/2)\mathbb{Z}_+} V/O''_{n,m}(V)$, which is clearly $(1/2)\mathbb{Z}_+$-graded with $M^{(m)}(n) = V/O''_{n,m}(V)$. For $u, v \in V$ and $p \in \mathbb{Z}$, set $d = n + \mathrm{wt}\, u - p - 1$, define a linear map u_p on $M(U)(n)$ mapping to $M(U)(d)$ by $u_p\left(v + O''_{n,m}(V)\right) = u *^d_{m,n} v + O''_{d,m}(V)$, if $d \geq 0$; and $u_p\left(v + O''_{n,m}(V)\right) = 0$ otherwise. By (3.2), we know $V *^n_{m,k} O''_{k,m}(V) \subseteq O''_{n,m}(V)$ for $k \in (1/2)\mathbb{Z}_+$. Thus, this action is well-defined. Then we form a generating function $Y_{M^{(m)}}(u,z) = \sum_{p \in \mathbb{Z}} u_p z^{-p-1}$.

Lemma 4.4. *Let $m \in (1/2)\mathbb{Z}_+$. Then*

(1) *for any $u \in V$ and $p \in \mathbb{Z}$, $u_p(M^{(m)}(n)) = 0$ if $p > \mathrm{wt}\, u + n - 1$;*
(2) *$Y_{M^{(m)}}(\mathbf{1}, z) = \mathrm{id}$;*
(3) *for any $a \in V^{\bar{r}}, b \in V^{\bar{s}}$ and $n \in (1/2)\mathbb{Z}_+$, we have*

$$(z_2+z_0)^{\mathrm{wt}\, a+q} Y_{M^{(m)}}\left(Y\left(a,z_0\right)b, z_2\right) = (z_0+z_2)^{\mathrm{wt}\, a+q} Y_{M^{(m)}}\left(a, z_0+z_2\right) Y_{M^{(m)}}\left(b, z_2\right)$$

or equivalently, for any $l \in \mathbb{Z}$,

$$\begin{aligned}&\mathrm{Res}_{z_0} z_0^l \left(z_2+z_0\right)^{\mathrm{wt}\, a+q} z_2^{\mathrm{wt}\, b-q} Y_{M^{(m)}}\left(Y\left(a,z_0\right)b, z_2\right)\\ =&\mathrm{Res}_{z_0} z_0^l \left(z_0+z_2\right)^{\mathrm{wt}\, a+q} z_2^{\mathrm{wt}\, b-q} Y_{M^{(m)}}\left(a, z_0+z_2\right) Y_{M^{(m)}}\left(b, z_2\right)\end{aligned}$$

on $M^{(m)}(n)$, where $q = -1 + \lfloor n \rfloor + \delta_{\hat{n}}(r) + r/2$.

Proof. (1) follows from the definition of u_p. And for (2), it is sufficient to show $\mathbf{1}_p = \delta_{p,-1}\,\mathrm{id}$ on $M^{(m)}(n)$ for any $n \in (1/2)\mathbb{Z}_+$. By (1), $\mathbf{1}_p = 0$ on $M^{(m)}(n)$ if $p > n-1$. Now considering $\mathbb{Z} \ni p \leq n-1$, then for any $v \in V$, set $d = \lfloor m \rfloor + \lfloor n-p-1 \rfloor - \lfloor n \rfloor$, we have

$$\begin{aligned}&\mathbf{1}_p(v + O''_{n,m}(V)) = \mathbf{1} *^{n-p-1}_{m,n} v + O''_{n-p-1,m}(V)\\ =&\sum_{i=0}^{\lfloor n \rfloor} (-1)^i \binom{d+i}{i} \mathrm{Res}_z \frac{(1+z)^{\lfloor m \rfloor}}{z^{d+i+1}} Y(\mathbf{1},z)v + O''_{n-p-1,m}(V)\\ =&\sum_{i=0}^{\lfloor n \rfloor} (-1)^i \binom{d+i}{i} \binom{\lfloor m \rfloor}{d+i} v + O''_{n-p-1,m}(V)\\ =&\sum_{i=0}^{\lfloor n \rfloor} (-1)^i \binom{\lfloor m \rfloor - p + i - 1}{i} \binom{\lfloor m \rfloor}{\lfloor m \rfloor - p + i - 1} v + O''_{n-p-1,m}(V)\\ =&\sum_{i=0}^{\lfloor n \rfloor} (-1)^i \binom{\lfloor m \rfloor}{p+1} \binom{p+1}{i} v + O''_{n-p-1,m}(V)\\ =&\delta_{p,-1} v + O''_{n-p-1,m}(V) \qquad \text{(by Notation 4.1 (1))}\\ =&\delta_{p,-1}(v + O''_{n,m}(V)).\end{aligned}$$

Thus, (2) holds. The idea of the proof of (3) comes essentially from [2, Lemma 5.10] (see also [6, Lemma 3.9]). For $v + O''_{n,m}(V) \in M^{(m)}(n)$, $q = -1 + \lfloor n \rfloor + \delta_{\hat{n}}(r) + r/2$ and let $\alpha \in \{0,1\}$ be such that $\bar{\alpha} = \overline{\hat{n} - r - s}$, we have

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_0^l (z_2+z_0)^{\mathrm{wt}\, a+q} z_2^{\mathrm{wt}\, b-q} Y_{M^{(m)}}(Y(a,z_0)b,z_2)(v+O''_{n,m}(V)) \\
&= \sum_{j\in\mathbb{Z}_+} \binom{\mathrm{wt}\, a+q}{j} z_2^{\mathrm{wt}\, a+\mathrm{wt}\, b-j} Y_{M^{(m)}}(a_{j+l}b,z_2)(v+O''_{n,m}(V)) \\
&= \sum_{j\in\mathbb{Z}_+} \binom{\mathrm{wt}\, a+q}{j} \sum_{k\in\alpha/2+\mathbb{Z}_+} z_2^{l+k-n+1} (a_{j+l}b)_{\mathrm{wt}\, a+\mathrm{wt}\, b-j-l-2-k+n}(v+O''_{n,m}(V)) \\
&= \sum_{k\in\alpha/2+\mathbb{Z}_+} z_2^{l+k-n+1} \sum_{j\in\mathbb{Z}_+} \binom{\mathrm{wt}\, a+q}{j} (a_{j+l}b) *_{m,n}^k v + O''_{k,m}(V) \\
&= \sum_{k\in\alpha/2+\mathbb{Z}_+} z_2^{l+k-n+1} \Big(\mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, a+q}}{z^{-l}} Y(a,z)b\Big) *_{m,n}^k v + O''_{k,m}(V) \\
&= \sum_{k\in\alpha/2+\mathbb{Z}_+} z_2^{l+k-n+1} \mathrm{Res}_z \left(f_{-l}(a,b) *_{m,n}^k v\right) + O''_{k,m}(V) \qquad \text{(by Notation 4.1(3))} \\
&= \sum_{k\in\alpha/2+\mathbb{Z}_+} z_2^{l+k-n+1} \sum_{j=0}^{k+1+q+l} (-1)^j \binom{l}{j} \sum_{i=0}^{k+1+q+l-j} (-1)^i \binom{-l+i+j-1}{i} \\
&\quad \times \mathrm{Res}_z \left(f_{i+j-l}(a,b) *_{m,n}^k v\right) + O''_{k,m}(V) \qquad \text{(by Notation 4.1 (2)-(3) and Lemma 4.2)} \\
&= \sum_{k\in\alpha/2+\mathbb{Z}_+} z_2^{l+k-n+1} \sum_{j=0}^{k+1+q+l} (-1)^j \binom{l}{j} \left(\left(a *_{n,k+1+q+l-j}^k b\right) *_{m,n}^k v\right) + O''_{k,m}(V) \\
&\hspace{20em} \text{(by Lemma 4.3)} \\
&= \sum_{\substack{k\in\alpha/2+\mathbb{Z}_+\\ k+1+l+q\geq 0}} z_2^{l+k-n+1} \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{l}{j} a *_{m,k+1+q+l-j}^k \left(b *_{m,n}^{k+1+q+l-j} v\right) + O''_{k,m}(V) \\
&\hspace{14em} \text{(by (3.2) and Notation 4.1 (1)-(2))} \\
&= \sum_{j\in\mathbb{Z}_+} \sum_{\substack{-n\leq i\in -s/2+\mathbb{Z}\\ -l+i+j\geq 1+q-n}} \binom{l}{j} (-1)^j z_2^{i+j-q} a *_{m,n+i}^{-l+i+j-1-q+n} \left(b *_{m,n}^{n+i} v\right) + O''_{-l+i+j-1-q+n,m}(V) \\
&= \sum_{j\in\mathbb{Z}_+} \binom{l}{j} (-1)^j a_{\mathrm{wt}\, a+q+l-j} \sum_{-n\leq i\in -s/2+\mathbb{Z}} z_2^{i+j-q} b_{\mathrm{wt}\, b-1-i}(v+O''_{n,m}(V)) \\
&= \sum_{j\in\mathbb{Z}_+} \binom{l}{j} (-1)^j a_{\mathrm{wt}\, a+q+l-j} z_2^{\mathrm{wt}\, b+j-q} Y_{M^{(m)}}(b,z_2)(v+O''_{n,m}(V)) \\
&= \mathrm{Res}_{z_0} z_0^l (z_0+z_2)^{\mathrm{wt}\, a+q} z_2^{\mathrm{wt}\, b-q} Y_{M^{(m)}}(a,z_0+z_2) Y_{M^{(m)}}(b,z_2)(v+O''_{n,m}(V)),
\end{aligned}$$

proving (3). □

As an immediate consequence of Lemma 4.4 and [12, Proposition 2.3.3], we have:

Proposition 4.5. *For any $m \in (1/2)\mathbb{Z}_+$, $M^{(m)}$ is an admissible V-module.*

For $u \in V^{\bar r}$ and $v \in V^{\bar s}$, if $\overline{\hat m - \hat n} = \overline{r+s}$, then it follows from [13] (see also [10]) that

$$Y(v,z)u \equiv (-1)^{\tilde u \tilde v}(1+z)^{-\operatorname{wt} u - \operatorname{wt} v - m + n} Y\left(u, \frac{-z}{1+z}\right) v \bmod L_{n,m}(V),$$

where $n, m \in (1/2)\mathbb{Z}_+$. From [10, Lemma 4.2 and Corollary 4.3], we have:

Lemma 4.6. *For $u \in V^{\bar r}$ and $v \in V^{\bar s}$, if $\overline{\hat p - \hat n} = \bar r, \overline{\hat m - \hat p} = \bar s$ and $m + n - p \geq 0$, then*

$$u *_{m,p}^{n} v - v *_{m,m+n-p}^{n} u - \operatorname{Res}_z (1+z)^{\operatorname{wt} u - 1 + p - n} Y(u,z)v \in L_{n,m}(V).$$

In particular, taking $p = m$ and $v = \mathbf{1}$ we have $u *_m^n \mathbf{1} - u \in L_{n,m}(V)$.

Theorem 4.7. *We have $O_{n,m}(V) = V^{\bar s} + L_{n,m}(V) + O''_{n,m}(V)$ with $\overline{\hat m - \hat n} \neq \bar s$.*

Proof. By Proposition 4.5 and Theorem 3.1 (2), $M^{(m)}(m) = V/O''_{m,m}(V) \subseteq \Omega_m(M^{(m)})$. Note that $O_{n,m}(V) = \left(O_{n,m}(V) \cap V^{\bar 0}\right) \bigoplus \left(O_{n,m}(V) \cap V^{\bar 1}\right)$ by $V = V^{\bar 0} \bigoplus V^{\bar 1}$ and $V^{\bar s} \subseteq O_{n,m}(V)$, where $\overline{\hat m - \hat n} \neq \bar s$. For any $u \in O_{n,m}(V) \cap V^{\bar r} = \mathcal{O}_{n,m}(V) \cap V^{\bar r}$ (see Theorem 3.2), then by the definition of $\mathcal{O}_{n,m}(V)$,

$$0 = o_{m-n}(u)(\mathbf{1} + O''_{m,m}(V)) = u *_m^n \mathbf{1} + O''_{n,m}(V),$$

that is $u *_m^n \mathbf{1} \in O''_{n,m}(V)$. If $\overline{\hat m - \hat n} = \bar r$, then by Lemma 4.6,

$$u = u - u *_m^n \mathbf{1} + u *_m^n \mathbf{1} \in L_{n,m}(V) + O''_{n,m}(V);$$

otherwise, $u \in V^{\bar s}$, where $\overline{\hat m - \hat n} \neq \bar s$. Thus by the definition of $O'_{n,m}(V)$, $O_{n,m}(V) = V^{\bar s} + L_{n,m}(V) + O''_{n,m}(V)$, where $\overline{\hat m - \hat n} \neq \bar s$. □

5. Universal enveloping algebra $U(V)$

In this section, we recall the universal enveloping algebra associated to SVOAs (cf. [6, Section 4]). Let V be a vertex operator superalgebra. Let $\hat V = L(V)/DL(V)$, where $L(V) = V \otimes \mathbb{C}\left[t, t^{-1}\right]$ and $D = 1 \otimes \frac{d}{dt} + L(-1) \otimes 1$. Denote by $a(m)$ the image of $a \otimes t^m \in L(V)$ in $\hat V$. For $a, b \in V$ and $m, k \in \mathbb{Z}$, define the Lie super-bracket as follows:

$$[a(m), b(k)] = \sum_{i=0}^{\infty} \binom{m}{i} (a_i b)(m + k - i).$$

Then $\hat V$ is a $(1/2)\mathbb{Z}$-graded Lie superalgebra with the degree of $a(m)$ defined to be $\operatorname{wt} a - m - 1$ for homogeneous $a \in V$. Let $U(\hat V)$ be the universal enveloping algebra of the Lie superalgebra $\hat V$. Then the $(1/2)\mathbb{Z}$-grading on $\hat V$ induces a $(1/2)\mathbb{Z}$-grading on $U(\hat V) = \bigoplus_{m \in (1/2)\mathbb{Z}} U(\hat V)_m$. Following from [6], we set

$$U(\hat V)_m^k = \sum_{(1/2)\mathbb{Z} \ni i \leq k} U(\hat V)_{m-i} U(\hat V)_i$$

for $(1/2)\mathbb{Z} \ni k < 0$ and $U(\hat V)_m^0 = U(\hat V)_m$, then $U(\hat V)_m^k \subseteq U(\hat V)_m^{k+1/2}$ and

$$\bigcap_{k \in -(1/2)\mathbb{Z}_+} U(\hat V)_m^k = 0, \qquad \bigcup_{k \in -(1/2)\mathbb{Z}_+} U(\hat V)_m^k = U(\hat V)_m.$$

Thus, $\left\{U(\hat{V})_m^k \mid k\in -(1/2)\mathbb{Z}_+\right\}$ forms a fundamental neighborhood system of $U(\hat{V})_m$. Let $\widetilde{U}(\hat{V})_m$ be the completions of $U(\hat{V})_m$, then $\tilde{U}(\hat{V})=\bigoplus_{m\in(1/2)\mathbb{Z}}\tilde{U}(\hat{V})_m$. For $m\in(1/2)\mathbb{Z}$, define a linear map $J_m(\cdot):V\to\hat{V}$ by $J_m(u)=u(\operatorname{wt}u+m-1)$. Note that $J_m(u)=0$ if $\operatorname{wt}u+m-1\notin\mathbb{Z}$.

Definition 5.1. The universal enveloping algebra $U(V)$ of V is the quotient of $\widetilde{U}(\hat{V})$ by the two-sided ideal generated by the relations: $\mathbf{1}(i)=\delta_{i,-1}$ for $i\in\mathbb{Z}$ and

$$\sum_{i\geq 0}(-1)^i\binom{l}{i}\left(J_{s-i}(u)J_{t+i}(v)-(-1)^{\tilde{u}\tilde{v}+l}J_{l+t-i}(v)J_{s+i-l}(u)\right)=\sum_{i\geq 0}\binom{d}{i}J_{s+t}\left(u_{l+i}v\right) \tag{5.1}$$

for any $u,v\in V$, $s\in(1/2)^{\tilde{u}}+\mathbb{Z}$, $t\in(1/2)^{\tilde{v}}+\mathbb{Z}$, $l\in\mathbb{Z}$, where $d=s+\operatorname{wt}u-l-1$.

Then $U(V)$ is also a $(1/2)\mathbb{Z}$-graded associative algebra $U(V)=\bigoplus_{m\in(1/2)\mathbb{Z}}U(V)_m$. Set

$$U(V)_m^k=\sum_{(1/2)\mathbb{Z}\ni i\leq k}U(V)_{m-i}U(V)_i$$

for any $(1/2)\mathbb{Z}\ni k<0$, then $U(V)_0/U(V)_0^k$ is an associative algebra, since $U(V)_0^k$ is a two-sided ideal of $U(V)_0$. Then $U(V)_{n-m}/U(V)_{n-m}^{-m-1/2}$ is a $U(V)_0/U(V)_0^{-n-1/2}-U(V)_0/U(V)_0^{-m-1/2}$-bimodule for $n,m\in(1/2)\mathbb{Z}_+$.

Remark 5.2. (1) From the construction of $U(V)$ we see that any weak V-module is naturally a $U(V)$-module with the action induced by the map $u(m)\mapsto u_m$ for any $u\in V$ and $m\in\mathbb{Z}$.

(2) In the following section we shall still use the same notation $J_s(u)$ to denote the image of $J_s(u)$ in $U(V)$ or its quotients.

6. Isomorphisms

By [10, Lemma 6.2], we can obtain the following result.

Lemma 6.1. *Let $u,v\in V$ and $m,n,p\in(1/2)\mathbb{Z}_+$. Then*

$$J_{m-n}\left(u*_{m,p}^n v\right)\equiv J_{p-n}(u)J_{m-p}(v)\quad \operatorname{mod} U(V)_{n-m}^{-m-1/2}.$$

Before stating the main result, we need to present two more lemmas.

Lemma 6.2. *For $u,v\in V$, $s\in(1/2)^{\tilde{u}}+\mathbb{Z}$, $t\in(1/2)^{\tilde{v}}+\mathbb{Z}$ and $n\in(1/2)\mathbb{Z}_+$, we have*

$$\begin{aligned}J_s(u)J_t(v)\equiv&-\sum_{i\geq 1}(-1)^i\binom{s-(1/2)^{\tilde{u}}-\lfloor n\rfloor-1}{i}J_{s-i}(u)J_{t+i}(v)\\&+\sum_{i\geq 0}\binom{(1/2)^{\tilde{u}}+\lfloor n\rfloor+\operatorname{wt}u}{i}J_{s+t}\left(u_{s+i-(1/2)^{\tilde{u}}-\lfloor n\rfloor-1}v\right)\operatorname{mod} U(V)_{-s-t}^{-n-1/2}.\end{aligned}$$

Proof. It follows from setting $l=s-k$ in (5.1), where $k=(1/2)^{\tilde{u}}+\lfloor n\rfloor+1$, that

$$\begin{aligned}J_s(u)J_t(v)=&-\sum_{i\geq 1}(-1)^i\binom{s-k}{i}J_{s-i}(u)J_{t+i}(v)\\&+(-1)^{\tilde{u}\tilde{v}}\sum_{i\geq 0}(-1)^{s+i-k}J_{s+t-k-i}(v)J_{k+i}(u)+\sum_{i\geq 0}\binom{k-1+\operatorname{wt}u}{i}J_{s+t}\left(u_{s+i-k}v\right).\end{aligned}$$

The lemma follows from the observation that the second term on the right hand side lies in $U(V)_{-s-t}^{-n-1/2}$. □

The following result generalizes [9, Lemma 3.1] (see also [7, Lemma 5.2]).

Lemma 6.3. *Let $n, m \in (1/2)\mathbb{Z}_+$, for any*

$$w = \sum J_{k_1}\left(u^1\right)\cdots J_{k_q}\left(u^q\right) \in U(V)_{n-m}/U(V)_{n-m}^{-m-1/2},$$

where $u^j \in V, k_j \in (1/2)^{\tilde{u^j}} + \mathbb{Z}$, there exists $u(w) \in V$ such that $w = J_{m-n}(u(w))$.

Proof. Without loss of generality, we may assume that $w = J_{k_1}\left(u^1\right)\cdots J_{k_q}\left(u^q\right)$. We proceed induction on $(q, m-k_q)$, called the *pattern* of w, to show the lemma. Assume that $q \geq 2$ and $k_q < m + 1/2$, since it is trivial if $q = 1$ or $k_q \geq m+1/2$. Write w as $J(q-2)J_s(u)J_t(v)$ with $J(q-2) = J_{k_1}\left(u^1\right)\cdots J_{k_{q-2}}\left(u^{q-2}\right), s = k_{q-1}, t = k_q$ and $u = u^{q-1}, v = u^q$. Then by Lemma 6.2,

$$\begin{aligned} w \equiv & -\sum_{i\geq 1}(-1)^i\binom{s-(1/2)^{\tilde{u}}-\lfloor m\rfloor-1}{i}J(q-2)J_{s-i}(u)J_{t+i}(v) \\ & + \sum_{i\geq 0}\binom{(1/2)^{\tilde{u}}+\lfloor m\rfloor+\mathrm{wt}\, u}{i}J(q-2)J_{s+t}\left(u_{s+i-(1/2)^{\tilde{u}}-\lfloor m\rfloor-1}v\right) \bmod U(V)_{n-m}^{-m-1/2}. \end{aligned}$$

Note that the pattern of each monomial on the right hand side is strictly less than $(q, m-k_q)$. So the lemma follows from the induction hypothesis. □

Theorem 6.4. *For any $n, m \in (1/2)\mathbb{Z}_+$, we define a linear map*

$$\varphi_{n,m} : A_{n,m}(V) \to U(V)_{n-m}/U(V)_{n-m}^{-m-1/2}$$

sending $u + O_{n,m}(V)$ to $J_{m-n}(u) + U(V)_{n-m}^{-m-1/2}$. Then $\varphi_{n,n}$ is an algebra isomorphism and $\varphi_{n,m}$ is an $A_n(V)-A_m(V)$-bimodule isomorphism.

Proof. We prove the theorem in three steps.

(Step 1) Show that $\varphi_{n,m}$ is well-defined. Recall from Theorem 4.7 that $O_{n,m}(V) = V^{\bar{s}} + L_{n,m}(V) + O''_{n,m}(V)$, where $\overline{\hat{m}-\hat{n}} \neq \bar{s}$. Then $J_{m-n}(V^{\bar{s}} + L_{n,m}(V)) = 0$ by the definition of $J_{m-n}(\cdot)$. By Lemma 6.1, we get $J_{m-n}(O''_{n,m}(V)) \equiv 0 \bmod U(V)_{n-m}^{-m-1/2}$. Thus, $J_{m-n}(O_{n,m}(V)) \subseteq U(V)_{n-m}^{-m-1/2}$.

(Step 2) Show that $\varphi_{n,m}$ is bijective. By Lemma 6.3, $\varphi_{n,m}$ is surjective. For $u \in V$, if $J_{m-n}(u) \in U(V)_{n-m}^{-m-1/2}$, then by Remark 5.2 (1), $o_{m-n}(u)|_{\Omega_m(M)} = 0$ for all weak V-modules M, so $u \in \mathcal{O}_{n,m}(V) = O_{n,m}(V)$ by Theorem 3.2. Thus $\varphi_{n,m}$ is injective.

(Step 3) Show that $\varphi_{n,n}$ is an algebra homomorphism and $\varphi_{n,m}$ is an $A_n(V)-A_m(V)$-bimodule homomorphism. For any $u, v \in V$,

$$\begin{aligned} & \varphi_{n,m}\left((u + O_{n,m}(V)) *_m^n (v + O_m(V))\right) \\ = & \varphi_{n,m}\left(u *_m^n v + O_{n,m}(V)\right) = J_{m-n}\left(u *_m^n v\right) + U(V)_{n-m}^{-m-1/2} \\ = & J_{m-n}(u)J_0(v) + U(V)_{n-m}^{-m-1/2} = \left(J_{m-n}(u) + U(V)_{n-m}^{-m-1/2}\right)\cdot\left(J_0(v) + U(V)_0^{-m-1/2}\right), \end{aligned}$$

where the third equality follows from Lemma 6.1. When $m = n$, $A_n(V) = A_{n,n}(V)$ by Theorem 3.2, we obtain that $\varphi_{n,n}$ is an algebra homomorphism. Then

$$\varphi_{n,m}\left((u+O_{n,m}(V)) *_m^n (v+O_m(V))\right) = \varphi_{n,m}\left(u+O_{n,m}(V)\right)\cdot\left(v+O_m(V)\right).$$

Thus, $\varphi_{n,m}$ is a right $A_m(V)$-module homomorphism. Similarly, $\varphi_{n,m}$ is a left $A_n(V)$-module homomorphism, completing the proof. □

In the proof of Theorem 6.4, **Step 1** does not rely on Theorem 4.7. We can directly prove that $\varphi_{n,m}$ is well-defined using $O_{n,m}(V) = O'_{n,m}(V) + O''_{n,m}(V) + O'''_{n,m}(V)$. When V is a vertex operator algebra, it was proved in [9] (see also [7]) that $A_n(V)$ and $U(V)_0/U(V)_0^{-n-1}$ are algebra isomorphic. Subsequently, in [6] (see also [8]), it was shown that $A_{n,m}(V)$ and $U(V)_{n-m}/U(V)_{n-m}^{-m-1}$ are bimodule isomorphic. In this paper, we achieve both of these results using a unified and simpler approach (see Theorem 6.4).

Declaration of competing interest

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