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Bimodules and associative algebras associated to SVOAs over an arbitrary field

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ABSTRACT

In [8,5,11], (generalized) Zhu's algebras are realized as subquotients of the universal enveloping algebras of vertex operator (super)algebras. In this paper, we provide a unified and concise proof of these results. As applications, we show that (generalized) Zhu's algebras [3,10] associated with vertex operator (super)algebras over an arbitrary algebraically closed field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$ can all be realized as subquotients of their universal enveloping algebras.

1. Introduction

Let V be a vertex operator algebra. Zhu [12] constructed the Zhu algebra $A(V)$ and proved that there is a one-to-one correspondence between the isomorphism classes of irreducible $A(V)$-modules and those of irreducible admissible V-modules. Following this idea of associating an associative algebra to a vertex operator algebra, Dong, Li, and Mason [2] constructed for each $n \in \mathbb{Z}_+$ an associative algebra $A_n(V)$, where $A_0(V) = A(V)$. Later, Dong and Jiang [1] developed a bimodule theory for vertex operator algebras: for any $n, m \in \mathbb{Z}_+$, they defined an $A_n(V) - A_m(V)$-bimodule $A_{n,m}(V)$. Using these bimodules, given any $A_m(V)$-module U, one can construct a Verma-type admissible V-module of the form $\bigoplus_{n\in\mathbb{Z}_+} A_{n,m}(V) \otimes_{A_m(V)} U$.

Recalling the definitions of $A_n(V)$ and $A_{n,m}(V)$, we know that $A_n(V)$ is the quotient space $V/O_n(V)$, and $A_{n,m}(V)$ is the quotient space $V/O_{n,m}(V)$. Define

$$\mathcal{O}_{n,m}(V) = \left\{u \in V \mid o_{m-n}(u)|_{\Omega_m(M)} = 0 \text{ for all weak } V\text{-modules } M\right\}$$

and set $\mathcal{O}_n(V) = \mathcal{O}_{n,n}(V)$. Han [5] showed that $O_n(V) = \mathcal{O}_n(V)$ and $O_{n,m}(V) = \mathcal{O}_{n,m}(V)$, thus providing a unified definition for $A_n(V)$ and $A_{n,m}(V)$.

The universal enveloping algebra $U(V)$ of a vertex operator algebra was introduced by Frenkel and Zhu [4]. It plays a fundamental role in the representation theory of vertex operator algebras, as every weak V-module naturally becomes a $U(V)$-module. Frenkel and Zhu [4] noted that the Zhu algebra $A(V)$ is isomorphic to a quotient of $U(V)_0$. It has been established in [8] (see also [6,4]) that $A_n(V)$ is a quotient algebra of $U(V)_0$ for any $n \in \mathbb{Z}_+$. Moreover, Han [5] showed that the $A_n(V) - A_m(V)$-bimodule $A_{n,m}(V)$ is a quotient of $U(V)_{n-m}$ for any $n, m \in \mathbb{Z}_+$.

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In our previous work with Han and Xiao [7] (see also [6]), the above results were extended to the setting of twisted representations of vertex operator algebras. The author [11] further generalized these results to vertex operator superalgebras using a more concise and unified approach.

In this paper, we study vertex operator superalgebras over an arbitrary field. Our results recover the main results of [8,5, 11], and our approach is both more concise and more unified. This paper is organized as follows. In Section 2, we recall the definition of a vertex operator superalgebra over an arbitrary field, together with its weak modules and admissible modules. In Section 3, we review the universal enveloping algebra of a vertex operator superalgebra and establish some isomorphism theorems. In Section 4, as applications, we realize (generalized) Zhu algebras associated with vertex operator (super)algebras over arbitrary fields as subquotients of their universal enveloping algebras.

2. Basics

In this section, we recall the definitions of vertex operator superalgebras, weak modules, and admissible modules over an arbitrary algebraically closed field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$. For any $k \in \mathbb{Z}$, let $\bar{k}$ denote its image in $\mathbb{Z}/2\mathbb{Z}$.

Definition 2.1. A *vertex operator superalgebra* is a 4-tuple $(V, Y, \mathbf{1}, \omega)$, where $V = \bigoplus_{n\in(1/2)\mathbb{Z}} V_n = V^{\bar{0}} \bigoplus V^{\bar{1}}$ is a $(1/2)\mathbb{Z}$-graded $\mathbb{F}$-vector space with $\dim V_n < \infty$ for all n and $V_n = 0$ for $n \ll 0$, where $V^{\bar{0}} = \bigoplus_{n\in\mathbb{Z}} V_n$ and $V^{\bar{1}} = \bigoplus_{n\in 1/2+\mathbb{Z}} V_n$. $\mathbf{1} \in V_0$, $\omega \in V_2$ and Y is a linear map from V to $\operatorname{End} V\left[\left[z, z^{-1}\right]\right]$ sending $u \in V$ to $Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$ satisfying the following axioms:

(1) $Y(\mathbf{1}, z) = \mathrm{id}_V$ and $u_n\mathbf{1} = \delta_{n,-1}u$ for any $n \geq -1$ and $u \in V$;
(2) $u_n v \in V^{\overline{i+j}}$ for any $u \in V^{\bar{i}}$, $v \in V^{\bar{j}}$ and $n \in \mathbb{Z}$; $u_m v \in V_{s+t-m-1}$ for $u \in V_s$, $v \in V_t$, $m \in \mathbb{Z}$ and $u_n v = 0$ for $n \gg 0$;
(3) the Virasoro algebra relations hold: $[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}c_V$ for $m, n \in \mathbb{Z}$, where $c_V \in \mathbb{F}$ and $L(m) = \omega_{m+1}$ for $m \in \mathbb{Z}$; $L(0)|_{V_m} = m\,\mathrm{id}_{V_m}$ for $m \in (1/2)\mathbb{Z}$ and $Y(L(-1)u, z) = \frac{d}{dz}Y(u, z)$ for $u \in V$;
(4) for any $u, v \in V$, $m, l, n \in \mathbb{Z}$, the *Jacobi identity* holds:

$$\sum_{i\geq 0}(-1)^i\binom{l}{i}\left(u_{m+l-i}v_{n+i} - (-1)^{\tilde{u}\tilde{v}}(-1)^l v_{n+l-i}u_{m+i}\right) = \sum_{i\geq 0}\binom{m}{i}\left(u_{l+i}v\right)_{m+n-i},$$

where $\tilde{x} = 0$ for $x \in V^{\bar{0}}$ and $\tilde{x} = 1$ for $x \in V^{\bar{1}}$. Whenever $\tilde{x}$ appears, we always assume that $x \in V^{\bar{0}}$ or $V^{\bar{1}}$.

For any $n \in (1/2)\mathbb{Z}$, elements in V_n are called *homogeneous*, and if $u \in V_n$, we define $\operatorname{wt} u = n$. When $\operatorname{wt} u$ appears, we always assume that u is homogeneous.

Definition 2.2. A *weak V-module* is a $\mathbb{F}$-vector space M equipped with a linear map from V to $\operatorname{End} M\left[\left[z, z^{-1}\right]\right]$ sending $u \in V$ to $Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$ satisfying the following axioms:

(1) $Y_M(\mathbf{1}, z) = \mathrm{id}_M$;
(2) for any $u \in V$, $w \in M$, $u_n w = 0$ for $n \gg 0$;
(3) for any $u, v \in V$, $m, l, n \in \mathbb{Z}$, the following *Jacobi identity* holds on M:

$$\sum_{i\geq 0}(-1)^i\binom{l}{i}\left(u_{m+l-i}v_{n+i} - (-1)^{\tilde{u}\tilde{v}}(-1)^l v_{n+l-i}u_{m+i}\right) = \sum_{i\geq 0}\binom{m}{i}\left(u_{l+i}v\right)_{m+n-i}. \tag{2.1}$$

Definition 2.3. An *admissible V-module* M is a weak V-module that carries a $(1/2)\mathbb{Z}_+$-grading $M = \bigoplus_{n\in(1/2)\mathbb{Z}_+} M(n)$ with $u_m M(n) \subseteq M(\operatorname{wt} u + n - m - 1)$ for any $u \in V$, $m \in \mathbb{Z}$ and $n \in (1/2)\mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of non-negative integers.

In the Definitions 2.1 and 2.3, when $V^{\bar{1}} = 0$ and $\bigoplus_{n\in 1/2+\mathbb{Z}_+} M(n) = 0$, we recover the notions of a vertex operator algebra V over $\mathbb{F}$, a weak V-module, and an admissible V-module over a vertex operator algebra.

3. Universal enveloping algebra $U(V)$ and isomorphisms

In this section, we recall the universal enveloping algebra associated to SVOAs [11] and establish some isomorphism theorems.

Let V be a vertex operator superalgebra over $\mathbb{F}$. Let $L(V) = V \otimes \mathbb{F}\left[t, t^{-1}\right]$, recall a Lie superalgebra $\hat{V} = L(V)/\mathcal{D}L(V)$ from [3] with the Lie super-bracket as follows:

$$[a(m),b(k)]=\sum_{i=0}^{\infty}\binom{m}{i}(a_ib)(m+k-i),$$

for any $a,b\in V$ and $m,k\in\mathbb{Z}$, where $a(m)$ the image of $a\otimes t^m\in L(V)$ in $\hat{V}$. Then $\hat{V}$ is a $(1/2)\mathbb{Z}$-graded Lie superalgebra with the degree of $a(m)$ defined to be $\mathrm{wt}\,a-m-1$ for $a\in V, m\in\mathbb{Z}$. Let $U(\hat{V})$ be the universal enveloping algebra of the Lie superalgebra $\hat{V}$. Then the $(1/2)\mathbb{Z}$-grading on $\hat{V}$ induces a $(1/2)\mathbb{Z}$-grading on $U(\hat{V})=\bigoplus_{m\in(1/2)\mathbb{Z}}U(\hat{V})_m$. Following from [11], we set

$$U(\hat{V})_m^k=\sum_{(1/2)\mathbb{Z}\ni i\le k}U(\hat{V})_{m-i}U(\hat{V})_i$$

for $(1/2)\mathbb{Z}\ni k<0$ and $U(\hat{V})_m^0=U(\hat{V})_m$, then $U(\hat{V})_m^k\subseteq U(\hat{V})_m^{k+1/2}$ and

$$\bigcap_{k\in-(1/2)\mathbb{Z}_+}U(\hat{V})_m^k=0,\qquad \bigcup_{k\in-(1/2)\mathbb{Z}_+}U(\hat{V})_m^k=U(\hat{V})_m.$$

Thus, $\left\{U(\hat{V})_m^k\mid k\in-(1/2)\mathbb{Z}_+\right\}$ forms a fundamental neighborhood system of $U(\hat{V})_m$. Let $\widetilde{U}(\hat{V})_m$ be the completions of $U(\hat{V})_m$ and set $\widetilde{U}(\hat{V})=\bigoplus_{m\in(1/2)\mathbb{Z}}\widetilde{U}(\hat{V})_m$. For $m\in(1/2)\mathbb{Z}$, define a linear map $J_m(\cdot):V\to\hat{V}$ by $J_m(u)=u(\mathrm{wt}\,u+m-1)$. Note that $J_m(u)=0$ if $\mathrm{wt}\,u+m-1\notin\mathbb{Z}$.

Definition 3.1. The universal enveloping algebra $U(V)$ of V is the quotient of $\widetilde{U}(\hat{V})$ by the two-sided ideal generated by the relations: $\mathbf{1}(i)=\delta_{i,-1}$ for $i\in\mathbb{Z}$ and

$$\sum_{i\ge0}(-1)^i\binom{l}{i}\left(J_{s-i}(u)J_{t+i}(v)-(-1)^{\tilde{u}\tilde{v}+l}J_{l+t-i}(v)J_{s+i-l}(u)\right)=\sum_{i\ge0}\binom{d}{i}J_{s+t}\left(u_{l+i}v\right)\tag{3.1}$$

for any $u,v\in V$, $s\in(1/2)^{\tilde{u}}+\mathbb{Z}$, $t\in(1/2)^{\tilde{v}}+\mathbb{Z}$, $l\in\mathbb{Z}$, where $d=s+\mathrm{wt}\,u-l-1$.

Then $U(V)$ is also a $(1/2)\mathbb{Z}$-graded associative algebra $U(V)=\bigoplus_{m\in(1/2)\mathbb{Z}}U(V)_m$. Set

$$U(V)_m^k=\sum_{(1/2)\mathbb{Z}\ni i\le k}U(V)_{m-i}U(V)_i$$

for any $(1/2)\mathbb{Z}\ni k<0$, then the quotient $U(V)_0/U(V)_0^k$ is an associative algebra, since $U(V)_0^k$ is two-sided ideal of $U(V)_0$. Then $U(V)_{n-m}/U(V)_{n-m}^{-m-1/2}$ is a $U(V)_0/U(V)_0^{-n-1/2}-U(V)_0/U(V)_0^{-m-1/2}$-bimodule for $n,m\in(1/2)\mathbb{Z}_+$.

Remark 3.2. (1) From the construction of $U(V)$, any weak V-module is naturally a $U(V)$-module with the action induced by the map $u(m)\mapsto u_m$ for any $u\in V$ and $m\in\mathbb{Z}$.
(2) We still use $J_s(u)$ to denote the image of $J_s(u)$ in $U(V)$ or its quotients.

For any $n\in(1/2)\mathbb{Z}$, there exists a unique $\hat{n}\in\{0,1\}$ such that $n=\lfloor n\rfloor+\hat{n}/2$, where $\lfloor\cdot\rfloor$ denotes the floor function. This decomposition is utilized whenever we refer to $n\in(1/2)\mathbb{Z}$. For $i,r\in\{0,1\}$, define $\delta_i(r)=1$, if $r\le i\le1$, and $\delta_i(r)=0$, if $i<r$ with the convention that $\delta_i(2)=0$. For $u\in V^{\bar{r}}(r=0,1)$, $v\in V$ and $n,m,p\in(1/2)\mathbb{Z}_+$, define the product $*^n_{m,p}$ on V as follows:

$$u*^n_{m,p}v=\sum_{j=0}^{\lfloor p\rfloor}(-1)^j\binom{\lfloor m\rfloor+\lfloor n\rfloor-\lfloor p\rfloor+\varepsilon+j}{j}\mathrm{Res}_z\frac{(1+z)^{\mathrm{wt}\,u+\lfloor m\rfloor+\delta_{\hat{m}}(r)-1+r/2}}{z^{\lfloor m\rfloor+\lfloor n\rfloor-\lfloor p\rfloor+\varepsilon+j+1}}Y(u,z)v,$$

if $\overline{\hat{p}-\hat{n}}=\bar{r}$, where $\varepsilon=-1+\delta_{\hat{m}}(r)+\delta_{\hat{n}}(2-r)$; and $u*^n_{m,p}v=0$ otherwise. Set $*^n_m=*^n_{m,m}$, $\bar{*}^n_m=*^n_{m,n}$ and $*_n=*^n_n=\bar{*}^n_n$. From [9, Lemma 6.2] and [11, Lemma 6.3], we have

Lemma 3.3. *(1) Let $u,v\in V$ and $m,n,p\in(1/2)\mathbb{Z}_+$. Then*

$$J_{m-n}\left(u*^n_{m,p}v\right)\equiv J_{p-n}(u)J_{m-p}(v)\ \mathrm{mod}\ U(V)_{n-m}^{-m-1/2}.$$

(2) For any $w=\sum J_{k_1}\left(u^1\right)\cdots J_{k_m}\left(u^m\right)\in U(V)_{n-m}/U(V)_{n-m}^{-n-1/2}$ with $n,m\in(1/2)\mathbb{Z}_+$, where $u^j\in V$, then there exists $u(w)\in V$ such that $w=J_{m-n}(u(w))$.

For $m\in(1/2)\mathbb{Z}_+$, set $\mathbb{M}^m=\bigoplus_{n\in(1/2)\mathbb{Z}_+}U(V)_{n-m}/U(V)_{n-m}^{-m-1/2}$. Then $\mathbb{M}^m$ carries a natural $(1/2)\mathbb{Z}_+$-grading such that $\mathbb{M}^m(n)=U(V)_{n-m}/U(V)_{n-m}^{-m-1/2}$ for $n\in(1/2)\mathbb{Z}_+$. For $u\in V,\ p\in\mathbb{Z}, n\in(1/2)\mathbb{Z}_+,\ v\in U(V)_{n-m}/U(V)_{n-m}^{-m-1/2}$, set $d=n+\mathrm{wt}\,u-p-1$, define a linear map u_p from $\mathbb{M}^m(n)$ to $\mathbb{M}^m(d)$ is defined by $u_p(v)=u(p)v$, if $d\geq 0$; otherwise $u_p(v)=0$. It is easy to verify that this map is well defined. Then we form a generating function $Y_{\mathbb{M}^m}(u,z)=\sum_{p\in\mathbb{Z}}u_pz^{-p-1}$. Then $\mathbb{M}^m$ carries the structure of an admissible V-module, since the Jacobi identity follows immediately from the construction of $U(V)$.

For any weak V-module M and $n,m\in(1/2)\mathbb{Z}_+$, let $o_n(\cdot)$ be the linear map from V to End M sending $u\in V$ to $o_n(u)=u_{\mathrm{wt}\,u+n-1}$ and define

$$\Omega_n(M)=\{w\in M\mid o_{n+i}(u)w=0 \text{ for all } u\in V \text{ and } 0<i\in(1/2)\mathbb{Z}\},$$

$$\mathcal{O}_{n,m}(V)=\left\{u\in V\ |o_{m-n}(u)|_{\Omega_m(M)}=0 \text{ for all weak } V\text{-modules } M\right\}.$$

Set $\mathcal{O}_n(V)=\mathcal{O}_{n,n}(V)$, $\mathcal{A}_n(V)=V/\mathcal{O}_n(V)$ and $\mathcal{A}_{n,m}=V/\mathcal{O}_{n,m}(V)$. Write $o(\cdot)=o_0(\cdot)$.

Lemma 3.4. *For $n,m\in(1/2)\mathbb{Z}_+$, define a linear map*

$$\varphi_{n,m}:\mathcal{A}_{n,m}(V)\to U(V)_{n-m}/U(V)_{n-m}^{-m-1/2}$$

sending $u+\mathcal{O}_{n,m}(V)$ to $J_{m-n}(u)+U(V)_{n-m}^{-m-1/2}$. Then $\varphi_{n,m}$ is a linear isomorphism.

Proof. Since $\mathbb{M}^m(m)=U(V)_0/U(V)_0^{-m-1/2}\subseteq\Omega_m(\mathbb{M}^m)$, for any $u\in\mathcal{O}_{n,m}(V)$, it follows from the definition of $\mathcal{O}_{n,m}(V)$ that $0=o_{m-n}(u)(J_0(\mathbf{1})+U(V)_0^{-m-1/2})=J_{m-n}(u)+U(V)_{n-m}^{-m-1/2}$. Hence, $\varphi_{n,m}$ is well-defined. Clearly, $\varphi_{n,m}$ is surjective by Lemma 3.3(2). Now suppose $J_{m-n}(u)\in U(V)_{n-m}^{-n-1/2}$. Then for any weak V-module M, we have $o_{m-n}(u)|_{\Omega_n(M)}=0$ by Remark 3.2(1). Therefore, $u\in\mathcal{O}_{n,m}(V)$ by definition, which implies that $\varphi_{n,m}$ is injective. Thus, the proof is complete. □

Using the fact that $U(V)_0/U(V)_0^{-n-1/2}$ is an associative algebra, $U(V)_{n-m}/U(V)_{n-m}^{-m-1/2}$ is a $U(V)_0/U(V)_0^{-n-1/2}-U(V)_0/U(V)_0^{-m-1/2}$-bimodule, and the linear isomorphisms in Lemma 3.4, we immediately obtain the following result.

Theorem 3.5. *Let $m,n\in(1/2)\mathbb{Z}_+$, then*
(1) *$\mathcal{A}_n(V)$ is an associative algebra under the multiplication $*_n$ with the identity $\mathbf{1}+\mathcal{O}_n(V)$.*
(2) *$\mathcal{A}_{n,m}(V)$ is an $\mathcal{A}_n(V)-\mathcal{A}_m(V)$-bimodule with $\bar{*}_m^n$ the left action and $*_m^n$ the right action.*

Theorem 3.6. *In Lemma 3.4, $\varphi_{n,n}$ is an algebra isomorphism and $\varphi_{n,m}$ is a bimodule isomorphism.*

Proof. For any $u,v\in V$,

$$\begin{aligned}&\varphi_{n,m}\left(\left(u+\mathcal{O}_{n,m}(V)\right)*_m^n(v+\mathcal{O}_m(V))\right)\\=&\varphi_{n,m}\left(u*_m^n v+\mathcal{O}_{n,m}(V)\right)=J_{m-n}\left(u*_m^n v\right)+U(V)_{n-m}^{-m-1/2}\\=&J_{m-n}(u)J_0(v)+U(V)_{n-m}^{-m-1/2}=\left(J_{m-n}(u)+U(V)_{n-m}^{-m-1/2}\right)\cdot\left(J_0(v)+U(V)_0^{-m-1/2}\right),\end{aligned}$$

where the third equality follows from Lemma 3.3(1). When $m=n$, $\mathcal{A}_n(V)=\mathcal{A}_{n,n}(V)$ by definition, we obtain that $\varphi_{n,n}$ is an algebra isomorphism. Then

$$\varphi_{n,m}\left(\left(u+\mathcal{O}_{n,m}(V)\right)*_m^n(v+\mathcal{O}_m(V))\right)=\varphi_{n,m}\left(u+\mathcal{O}_{n,m}(V)\right)\cdot(v+\mathcal{O}_m(V)).$$

Thus, $\varphi_{n,m}$ is a right $\mathcal{A}_m(V)$-module homomorphism. Similarly, $\varphi_{n,m}$ is a left $\mathcal{A}_n(V)$-module homomorphism, completing the proof. □

4. Applications

In this section, as applications, we realize (generalized) Zhu algebras associated with vertex operator (super)algebras over arbitrary fields as subquotients of their universal enveloping algebras.

4.1. Associative algebra $A(V)$ associated SVOAs over $\mathbb{F}$

For a vertex operator superalgebra V over $\mathbb{F}$, any $a, b \in V$ and any $m, n \in \mathbb{Z}_+$, we set

$$a * b = \begin{cases} a *_0 b, & \text{if } a, b \in V^{\bar{0}}, \\ 0, & \text{if } a \text{ or } b \in V^{\bar{1}}, \end{cases} \qquad a \circ_m^n b = \begin{cases} \operatorname{Res}_z Y(a, z) \frac{(1+z)^{\mathrm{wt}\, a + m}}{z^{2+n+m}} b, & \text{if } a \in V^{\bar{0}}, \\ \operatorname{Res}_z Y(a, z) \frac{(1+z)^{\mathrm{wt}\, a - 1/2 + m}}{z^{1+n+m}} b, & \text{if } a \in V^{\bar{1}}, \end{cases}$$

and extend to $V \times V$ bilinearly. Denote by $O(V)$ the linear span of $a \circ_m^n b$ for $a, b \in V, m, n \in \mathbb{Z}_+$ and set $A(V) = V/O(V)$. From [3], we have

Theorem 4.1. (1) *$A(V)$ is an associative algebra under $*$ with the identity $\mathbf{1} + O(V)$.*

(2) *Suppose that M is a weak V-module. Then there is a representation of the associative algebra $A(V)$ on $\Omega_0(M)$ induced by the map $u \mapsto o(u)$ for $u \in V$. Let $M = \bigoplus_{k \in (1/2)\mathbb{Z}_+} M(k)$ be an admissible V-module, then $M(0)$ is $A(V)$-module under $u + O(V) \mapsto o(u)$.*

(3) *There exists an admissible V-module $\bar{M}(A(V))$ such that $\bar{M}(A(V))(0) = A(V)$.*

Lemma 4.2. *We have $O(V) = \mathcal{O}_0(V)$.*

Proof. For any weak V-module M, $\Omega_0(M)$ is an $A(V)$-module, so for any $u \in O(V)$, we have $o(u) = 0$ on $\Omega_0(M)$. Thus $O(V) \subseteq \mathcal{O}_0(V)$. Consider admissible V-module $\bar{M}(A(V))$, so $\bar{M}(A(V))(0) = A(V) \subseteq \Omega_0(\bar{M}(A(V)))$. For any $u \in \mathcal{O}_0(V)$, we have

$$0 = o(u)(\mathbf{1} + O(V)) = u * \mathbf{1} + O(V) = u + O(V).$$

Thus $\mathcal{O}_0(V) \subseteq O(V)$. $\square$

From Theorem 3.6 and Lemma 4.2, we have

Corollary 4.3. *There exists an algebra isomorphism $A(V) \cong U(V)_0/U(V)_0^{-1/2}$.*

4.2. Associative algebra $A_n(V)$ associated VOAs over $\mathbb{F}$

Let V be a vertex operator algebra over $\mathbb{F}$. For $n \in \mathbb{Z}_+$, let $O_n(V)$ be the linear span of all $a \circ_{n,t}^s b$ and $L(-1)a + L(0)a$ where for $a \in V$ and $b \in V$,

$$a \circ_{n,t}^s b = \operatorname{Res}_z Y(a, z) b \frac{(1+z)^{\mathrm{wt}\, a + n + s}}{z^{2n+2+t}}$$

and $s, t \in \mathbb{Z}_+$ with $s \le t$. Define $A_n(V) = V/O_n(V)$. From [10], we have

Theorem 4.4. (1) *$A_n(V)$ is an associative algebra under $*_n$ with the identity $\mathbf{1} + O_n(V)$.*

(2) *Suppose that M is a weak V-module. Then there is a representation of the associative algebra $A_n(V)$ on $\Omega_n(M)$ induced by the map $u \mapsto o(u)$ for $u \in V$. Let $M = \bigoplus_{k \in \mathbb{Z}_+} M(k)$ be an admissible V-module, then $M(k)$ is $A_n(V)$-module under $u + O_n(V) \mapsto o(u)$ for $0 \le k \le n$.*

(3) *There exists an admissible V-module $\bar{M}(A_n(V))$ such that $\bar{M}(A_n(V))(n) = A_n(V)$.*

Lemma 4.5. *We have $O_n(V) = \mathcal{O}_n(V)$ for any $n \in \mathbb{Z}_+$.*

Proof. For any weak V-module M, $\Omega_n(M)$ is an $A_n(V)$-module, so for any $u \in O_n(V)$, we have $o(u) = 0$ on $\Omega_n(M)$. Thus $O_n(V) \subseteq \mathcal{O}_n(V)$. Consider admissible V-module $\bar{M}(A_n(V))$, so $\bar{M}(A_n(V))(n) = A_n(V) \subseteq \Omega_n(\bar{M}(A_n(V)))$. For any $u \in \mathcal{O}_n(V)$, we have

$$0 = o(u)(\mathbf{1} + O_n(V)) = u *_n \mathbf{1} + O_n(V) = u + O_n(V).$$

Thus $\mathcal{O}_n(V) \subseteq O_n(V)$. $\square$

For a vertex operator algebra V, its universal enveloping algebra $U(V)$ is $\mathbb{Z}$-graded. Therefore, from Theorem 3.6 and Lemma 4.5, we obtain the following result:

Corollary 4.6. *There exists an algebra isomorphism $A_n(V) \cong U(V)_0/U(V)_0^{-n-1}$.*

4.3. Associative algebra $A_n(V)$ and $A_n(V)-A_m(V)$-bimodules $A_{n,m}(V)$ associated SVOAs over $\mathbb{C}$

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator superalgebra over $\mathbb{C}$. Let $m, n \in (1/2)\mathbb{Z}_+$, define $O'_{n,m}(V) = V^{\bar{r}} + \mathrm{span}\{u \circ_m^n v \mid u, v \in V\} + L_{n,m}(V)$, where $\overline{\hat{m}-\hat{n}} \neq \bar{r}$, $L_{n,m}(V) = \mathrm{span}\{(L(-1)+L(0)+m-n)u \mid u \in V^{\bar{s}}$ such that $\overline{\hat{m}-\hat{n}} = \bar{s}\}$ and for $u, v \in V$,

$$u \circ_m^n v = \begin{cases} \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, u + \lfloor m \rfloor}}{z^{\lfloor m \rfloor + \lfloor n \rfloor + 2}} Y(u,z)v, & \text{if } u \in V^{\bar{0}}, \\ \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, u + \lfloor m \rfloor + \delta_{\hat{m}}(1) - 1/2}}{z^{\lfloor m \rfloor + \lfloor n \rfloor + \delta_{\hat{m}}(1) + \delta_{\hat{n}}(1) + 1}} Y(u,z)v, & \text{if } u \in V^{\bar{1}}. \end{cases}$$

Set $O_n(V) = O'_{n,n}(V)$ and $A_n(V) = V/O_n(V)$. For $u, a, b, c \in V$ and $p_1, p_2, p_3 \in (1/2)\mathbb{Z}_+$, we define $O''_{n,m}(V)$ as the linear span of $u *_{m,p_3}^n \left((a *_{p_1,p_2}^{p_3} b) *_{m,p_1}^{p_3} c - a *_{m,p_2}^{p_3} \left(b *_{m,p_1}^{p_2} c\right)\right)$. Define $O'''_{n,m}(V) = \sum_{p_1,p_2 \in (1/2)\mathbb{Z}_+} \left(V *_{p_1,p_2}^n O'_{p_2,p_1}(V)\right) *_{m,p_1}^n V$, $O_{n,m}(V) = O'_{n,m}(V) + O''_{n,m}(V) + O'''_{n,m}(V)$ and $A_{n,m}(V) = V/O_{n,m}(V)$. The subsequent theorem is derived from [9,11].

Theorem 4.7. *Let $m, n \in (1/2)\mathbb{Z}_+$, then*

(1) *$A_n(V)$ is an associative algebra under the multiplication $*_n$ with the identity $\mathbf{1} + O_n(V)$.*
(2) *$A_{n,m}(V)$ is an $A_n(V)-A_m(V)$-bimodule with $\bar{*}_m^n$ the left action and $*_m^n$ the right action.*
(3) *$O_n(V) = \mathcal{O}_n(V)$ and $O_{n,m}(V) = \mathcal{O}_{n,m}(V)$.*

From Theorem 3.6 and Theorem 4.7, we obtain the following result:

Corollary 4.8. *There exist an algebra isomorphism $A_n(V) \cong U(V)_0/U(V)_0^{-n-1/2}$ and a bimodule isomorphism $A_{n,m}(V) \cong U(V)_{n-m}/U(V)_{n-m}^{-n-1/2}$.*

Remark 4.9. Corollary 4.8 was first obtained in [11].

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Data availability

No data was used for the research described in the article.

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