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Research Paper

Twisted bimodules and universal enveloping algebras associated to VOAs

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ABSTRACT

For any vertex operator algebra V, finite automorphism g of V of order T and $m,n \in (1/T)\mathbb{Z}_+$, we construct a family of associative algebras $\mathcal{A}_{g,n}(V)$ and $\mathcal{A}_{g,n}(V)-\mathcal{A}_{g,m}(V)$-bimodules $\mathcal{A}_{g,n,m}(V)$ from the point of view of representation theory. We prove that the algebra $\mathcal{A}_{g,n}(V)$ is identical to the algebra $A_{g,n}(V)$ constructed by Dong, Li and Mason, and that the bimodule $\mathcal{A}_{g,n,m}(V)$ is identical to $A_{g,n,m}(V)$ which was constructed by Dong and Jiang. We also prove that the $A_{g,n}(V)-A_{g,m}(V)$-bimodule $A_{g,n,m}(V)$ is isomorphic to $U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}$, where $U(V[g])_k$ is the subspace of degree k of the $(1/T)\mathbb{Z}$-graded universal enveloping algebra $U(V[g])$ of V with respect to g and $U(V[g])_k^l$ is some subspace of $U(V[g])_k$. And we show that all these bimodules $A_{g,n,m}(V)$ can be defined in a simpler way.

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1. Introduction

The representation theory of vertex operator algebras is quiet different from that of classical algebras because of the appearance of twisted modules. Among all representations of a vertex operator algebra, admissible twisted modules are the most important ones. Recall that for a vertex operator algebra V and an automorphism g of V of finite order T, an admissible g-twisted V-module M is $(1/T)\mathbb{Z}_+$-graded: $M = \bigoplus_{i\in(1/T)\mathbb{Z}_+} M_i$ (cf. [7]). Thus, in order to study M it is vital to determine all $\mathrm{Hom}(M_i, M_j)$ for $i, j \in (1/T)\mathbb{Z}_+$. In fact, a series of associative algebras $A_{g,n}(V)$ was introduced (see [7,8]) for which there is an algebra homomorphism from $A_{g,n}(V)$ to $\mathrm{Hom}(M_i, M_i)$ for any $i \le n$. And in generally, to study these $\mathrm{Hom}(M_i, M_j)$, a series of $A_{g,n}(V)-A_{g,m}(V)$-bimodules $A_{g,n,m}(V)$ for $m, n \in (1/T)\mathbb{Z}_+$ was constructed by Dong and Jiang in [4], for which there is an $A_{g,n}(V)-A_{g,m}(V)$-bimodule homomorphism from $A_{g,n,m}(V)$ to $\mathrm{Hom}(M_{m-l}, M_{n-l})$ for any $0 \le l \le \min\{m, n\}$. Thus, in this sense bimodules $A_{g,n,m}(V)$ are generalizations of these associative algebras $A_{g,n}(V)$ (see [4]).

There are several kinds of associative algebras associated to vertex operator (super)algebras (see [2,6,9,14,18,20,22,23,28]); for the twisted case one can refer to [10,11,19,24,25].

From the construction, the bimodule $A_{g,n,m}(V)$ is the quotient of V by $O_{g,n,m}(V)$. Thus, for better understanding how these $A_{g,n,m}(V)$ can be used to study admissible twisted modules, a key step is to study the subspace $O_{g,n,m}(V)$. Intuitively, $O_{g,n,m}(V)$ should be closely related to twisted modules. This is indeed the case when $g = 1$. It was proved in [15] that $O_{1,n,m}(V)$ can also be defined from representations of V. This drives us to do so for general automorphisms. In fact we shall consider another subspace $\mathcal{O}_{g,n,m}(V)$ from the perspective of representation theory and define $\mathcal{A}_{g,n,m}(V)$ as the quotient of V by $\mathcal{O}_{g,n,m}(V)$. In this way, we obtain a series of associative algebras $\mathcal{A}_{g,n}(V)$. As a result, $\mathcal{A}_{g,n,m}(V)$ becomes an $\mathcal{A}_{g,n}(V)-\mathcal{A}_{g,m}(V)$-bimodule (see Theorem 3.5). And for any $\mathcal{A}_{g,m}(V)$-module U, we use a different way from [15] to show that $\bigoplus_{n\in(1/T)\mathbb{Z}_+} \mathcal{A}_{g,n,m}(V) \otimes_{\mathcal{A}_{g,m}(V)} U$ is an admissible g-twisted modules with some universal property (see Theorem 3.7). Based on this universal property we can show that $\mathcal{A}_{g,n,m}(V)$ is identical to $A_{g,n,m}(V)$ (see Theorem 5.2). Thus, all bimodules can be reconstructed from the perspective of representation theory. A generalization of twisted bimodules is constructed in [27] also from this perspective.

It is well known that the most powerful tool in the representation theory of Lie algebras is the universal enveloping algebra. As for a vertex operator algebra V and an automorphism g, a weak version of such universal enveloping algebra $U(V[g])$ also exists: every weak g-twisted V-module is automatically a $U(V[g])$-module. In fact, these universal enveloping algebras have close connection with $A_{g,n}(V)$ and bimodules $A_{g,n,m}(V)$. Frenkel and Zhu [14] pointed out that Zhu's algebra can be identified with some quotient of $U(V[1])$. It was proved in [17] that all $A_{g,n}(V)$ for $n \in \mathbb{Z}_+$ are some quotients of the universal enveloping algebra $U(V[g])$ for the case $g = 1$ and in [16] for the general finite automorphism g. Furthermore, the bimodule $A_{1,n,m}(V)$ was also proved to be some quotient of $U(V[1])$ (see [15]). In this present paper, we are going to consider the general g

and show that the $A_{g,n}(V)$–$A_{g,m}(V)$-bimodule $A_{g,n,m}(V)$ is some quotient of $U(V[g])$ (see Theorem 5.2).

As mentioned above, the bimodule $A_{g,n,m}(V)$ is the quotient of V by $O_{g,n,m}(V)$. For some technical reason, $O_{g,n,m}(V)$ is defined as the sum of three subspaces $O'_{g,n,m}(V)$, $O''_{g,n,m}(V)$ and $O'''_{g,n,m}(V)$. But it was conjectured that $O_{g,n,m}(V) = O'_{g,n,m}(V)$ (see [3]), that is, $O''_{g,n,m}(V)$ and $O'''_{g,n,m}(V)$ are superfluous. In this present paper we shall approach to this conjecture. More precisely, we shall show that $O'''_{g,n,m}(V)$ is superfluous and $O'_{g,n,m}(V)$ can be replaced by its subspace $\bigoplus_{s\not\equiv \bar{m}-\bar{n} \bmod T} V^s + L_{n,m}(V)$, or equivalently,

$$O_{g,n,m}(V) = \bigoplus_{s\not\equiv \bar{m}-\bar{n} \bmod T} V^s + L_{n,m}(V) + O''_{g,n,m}(V) \quad \text{(see Theorem 6.8).}$$

In this way, we can refine all these bimodules $A_{g,n,m}(V)$.

The paper is organized as follows: In Section 2, we recall the definition of $A_{g,n}(V)$–$A_{g,m}(V)$-bimodule $A_{g,n,m}(V)$ and the construction of the universal admissible g-twisted V-module $M(U)$. In Section 3, we construct the associative algebras $\mathcal{A}_{g,n}(V)$, the $\mathcal{A}_{g,n}(V)$–$\mathcal{A}_{g,m}(V)$-bimodules $\mathcal{A}_{g,n,m}(V)$ and use these bimodules $\mathcal{A}_{g,n,m}(V)$ to construct admissible g-twisted V-modules $\mathcal{M}(U)$ which have some universal property. In Section 4, we first recall the definition of universal enveloping algebra $U(V[g])$ of a vertex operator algebra V with respect to a finite automorphism g. Then we obtain another $A_{g,n}(V)$–$A_{g,m}(V)$-bimodule $U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}$ and construct universal admissible g-twisted V-modules $\mathbb{M}(U)$. In Section 5, we prove that $A_{g,n}(V)$ and $\mathcal{A}_{g,n}(V)$ are identical as associative algebras; $A_{g,n,m}(V)$, $\mathcal{A}_{g,n,m}(V)$ and $U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}$ are isomorphic to each other as $A_{g,n}(V)$–$A_{g,m}(V)$-bimodules. The Section 6 is devoted to refining the definition of the bimodules $A_{g,n,m}(V)$.

We assume that the reader is familiar with the basic knowledge on the vertex operator algebra theory such as the definition of vertex operator algebra (cf. [1], [5], [12], [13], [21]) and the definitions of weak and admissible twisted modules (cf. [7,8]).

2. $A_{g,n}(V)$–$A_{g,m}(V)$-bimodule $A_{g,n,m}(V)$

Let $V = \oplus_{n\in\mathbb{Z}} V_n$ be a vertex operator algebra and g an automorphism of V of finite order T. Then we have the decomposition $V = \bigoplus_{r=0}^{T-1} V^r$, where $V^r = \{v \in V \mid gv = e^{-2\pi ri/T}v\}$ (here and only here i represents the imaginary unit). For any n, elements u in V_n are called homogenous and we define $\mathrm{wt}\, u = n$. So when $\mathrm{wt}\, u$ appears we always assume that u is homogenous.

Let M be a weak g-twisted V-module. Recall from [7] (see also [13]) that for $u \in V^r$, $v \in V$ and $w \in M$, the twisted Jacobi identity

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y_M(u,z_1)\, Y_M(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y_M(v,z_2)\, Y_M(u,z_1) \\ &= z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-r/T}\delta\left(\frac{z_1-z_0}{z_2}\right) Y_M\left(Y(u,z_0)\, v, z_2\right) \end{aligned} \tag{2.1}$$

is equivalent to the weak associativity

$$(z_0+z_2)^{l+\frac{r}{T}}\, Y_M\left(u, z_0+z_2\right) Y_M\left(v, z_2\right) w=\left(z_2+z_0\right)^{l+\frac{r}{T}}\, Y_M\left(Y\left(u, z_0\right) v, z_2\right) w \tag{2.2}$$

where l is a nonnegative integer, and the commutator formula

$$\left[Y_M\left(u, z_1\right), Y_M\left(v, z_2\right)\right]=\operatorname{Res}_{z_0} z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1-z_0}{z_2}\right) Y_M\left(Y\left(u, z_0\right) v, z_2\right). \tag{2.3}$$

In fact, the twisted Jacobi identity can be replaced directly by the weak associativity (cf. [21,26]). The following theorem is stated in [26], the detailed proof of which is left in the appendix of the present paper.

Theorem 2.1. *[26] Let $(V, Y, \mathbf{1})$ be a vertex algebra and T a positive integer. Let M be a vector space and let $Y_M(\cdot, x)$ be a linear map from V to $(\operatorname{End} M)[[x^{\frac{1}{T}}, x^{-\frac{1}{T}}]]$ such that $Y_M(\mathbf{1}, x)=\mathrm{id}_M$ and $Y_M(v, x)w \in M((x^{\frac{1}{T}}))$ for $v \in V$ and $w \in M$. Set*

$$V^r=\left\{v \in V \mid Y_M(v, x) w \in x^{-\frac{r}{T}} M((x)) \text{ for any } w \in M\right\}$$

for $r \in \mathbb{Z}$. Then the twisted Jacobi identity (2.1) for $u \in V^r, v \in V$ and $w \in M$ is equivalent to the weak associativity (2.2).

For any $n \in (1/T)\mathbb{Z}$, define a linear map $o_n(\cdot): V \rightarrow \operatorname{End} M$ sending each element $v \in V$ to $v_{\mathrm{wt}\, v-1+n}$ and by [8] also define

$$\begin{aligned}
\Omega_n(M) & =\left\{w \in M \mid v_{\mathrm{wt}\, v-1+i} w=0 \text{ for all } v \in V \text{ and } n<i \in(1/T) \mathbb{Z}\right\} \\
& =\left\{w \in M \mid o_{n+i}(v) w=0 \text{ for all } v \in V \text{ and } 0<i \in(1/T) \mathbb{Z}\right\}.
\end{aligned}$$

For short, write $o_n(\cdot)$ as $o(\cdot)$ if $n=0$. And in this paper, for any two formal variables x, y and any $\alpha \in \mathbb{R}$, we define

$$(x+y)^\alpha=\sum_{i \in \mathbb{Z}_{+}} \frac{\alpha(\alpha-1) \cdots(\alpha-i+1)}{i!} y^i x^{\alpha-i}.$$

Then one can see that

$$(x+y)^\alpha(x+y)^\beta=(x+y)^{\alpha+\beta} \quad \text{for any } \alpha, \beta \in \mathbb{R}.$$

Proposition 2.2. *Let W be a weak g-twisted V-module, $u \in V^r, v \in V^s, p \in r/T+\mathbb{Z}, q \in s/T+\mathbb{Z}$ and $w \in W$. Let $l \in \mathbb{Z}_{+}$ be such that*

$$u_n w=0 \quad \text{for } n \geq l+r/T,$$

and $k \in \mathbb{Z}_+$ such that

$$v_n w = 0 \quad \textit{for } n > k + q.$$

And also let $t \in \mathbb{Z}$ be such that

$$u_n v = 0 \quad \textit{for} \quad n > t.$$

Then

$$u_p(v_q w) = \sum_{i=0}^{k} \sum_{j=0}^{N} \binom{p - l - r/T}{i} \binom{l + r/T}{j} (u_{p-l-r/T-i+j} v)_{q+l+r/T+i-j} w,$$

where $N = \max\{t - p + l + r/T + k, 0\}$.

Proof. By (2.2), $z_1^p z_2^q Y(u, z_1) Y(v, z_2) w \in W[[z_1, z_1{}^{-1}, z_2, z_2{}^{-1}]]$ and following the proof of [21, Proposition 4.5.7] we see that

$$\begin{aligned}
&u_p(v_q w) \\
&= \mathrm{Res}_{z_1} \mathrm{Res}_{z_2} z_1^p z_2^q Y(u, z_1) Y(v, z_2) w \\
&= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} (z_0 + z_2)^p z_2^q Y(u, z_0 + z_2) Y(v, z_2) w \\
&= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} (z_0 + z_2)^{p-l-r/T} z_2^q \big((z_0 + z_2)^{l+r/T} Y(u, z_0 + z_2) Y(v, z_2) w\big) \\
&= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} (z_0 + z_2)^{p-l-r/T} z_2^q \big((z_2 + z_0)^{l+r/T} Y(Y(u, z_0) v, z_2) w\big) \\
&= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} \sum_{i=0}^{k} \binom{p - l - r/T}{i} z_0^{p-l-r/T-i} z_2^{i+q} \big((z_2 + z_0)^{l+r/T} Y(Y(u, z_0) v, z_2) w\big),
\end{aligned}$$

which immediately gives the desired formula. $\square$

As for these $\Omega_m(M)$ we have:

Lemma 2.3. *$o_n(v)\Omega_m(M) \subset \Omega_{m-n}(M)$ for $m, n \in (1/T)\mathbb{Z}$ and $v \in V$.*

Proof. Without loss of generality we may assume that $v \in V^s$. Take any $u \in V^r$. Then for $0 < i \in (1/T)\mathbb{Z}$ and $w \in \Omega_m(M)$, according to Proposition 2.2, we can choose proper $k, l, N \in \mathbb{Z}_+$ and obtain

$$\begin{aligned}
&o_{m-n+i}(u) o_n(v) w = u_{\mathrm{wt}\, u+m-n+i-1}(v_{\mathrm{wt}\, v+n-1} w) \\
&= \sum_{p=0}^{k} \sum_{q=0}^{N} \binom{\mathrm{wt}\, u + m - n + i - 1 - l - r/T}{p} \binom{l + r/T}{q} \\
&\qquad \times (u_{\mathrm{wt}\, u+m-n+i-1-l-r/T-p+q} v)_{\mathrm{wt}\, v+n-1+l+r/T+p-q} w
\end{aligned}$$

$$= \sum_{p=0}^{k} \sum_{q=0}^{N} \binom{\mathrm{wt}\, u + m - n + i - l - r/T}{p} \binom{l + r/T}{q} o_{m+i}(u_{\mathrm{wt}\, u + t - n + i - 1 - l - r/T - p + q} v) w$$
$$= 0,$$

proving the lemma. □

For any $n \in (1/T)\mathbb{Z}_+$, there exists an $\bar{n} \in \{0, 1, \cdots, T-1\}$ such that $n = \lfloor n \rfloor + \bar{n}/T$, where $\lfloor \cdot \rfloor$ is the floor function. For $0 \le r \le T-1$, define $\delta_i(r) = 1$ if $r \le i \le T-1$ and $\delta_i(r) = 0$ if $i < r$; and set $\delta_i(T) = 0$.

Now for $u \in V^r, v \in V$ and $m, n, p \in (1/T)\mathbb{Z}$, define the product $*_{g,m,p}^n$ on V as follows:

$$u *_{g,m,p}^n v = \sum_{i=0}^{\lfloor p \rfloor} (-1)^i \binom{\lfloor m \rfloor + \lfloor n \rfloor - \lfloor p \rfloor - 1 + \delta_{\bar{m}}(r) + \delta_{\bar{n}}(T-r) + i}{i}$$
$$\cdot \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, u - 1 + \lfloor m \rfloor + \delta_{\bar{m}}(r) + r/T}}{z^{\lfloor m \rfloor + \lfloor n \rfloor - \lfloor p \rfloor + \delta_{\bar{m}}(r) + \delta_{\bar{n}}(T-r) + i}} Y(u, z) v,$$

if $m, n, p \in (1/T)\mathbb{Z}_+$ and $\bar{p} - \bar{n} \equiv r \mod T$; and $u *_{g,m,p}^n v = 0$ otherwise. Denote $*_{g,m,p}^n$ by $\bar{*}_{g,m}^n$ if $p = n$ and by $*_{g,m}^n$ if $p = m$. In particular,

$$\mathbf{1} \bar{*}_{g,m}^n u = u \quad \text{for } u \in V. \tag{2.4}$$

Note that if $g = 1$, $*_{1,m,p}^n$ is the same as $*_{m,p}^n$ defined in [3]. And $*_{g,n,n}^n$ is, in fact, the product $*_{g,n}$ defined in [8]; in particular, $*_{g,0}$ is the product $*_g$ defined in [7].

For $m, n \in (1/T)\mathbb{Z}_+$, let

$$O'_{g,n,m}(V) = \mathrm{span}\{u \circ_{g,m}^n v \mid u, v \in V\} + L_{n,m}(V),$$

where $L_{n,m}(V) = \mathrm{span}\{(L(-1) + L(0) + m - n)u \mid u \in V\}$ and for $u \in V^r$, $v \in V$,

$$u \circ_{g,m}^n v = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, u - 1 + \delta_{\bar{m}}(r) + \lfloor m \rfloor + r/T}}{z^{\lfloor m \rfloor + \lfloor n \rfloor + \delta_{\bar{m}}(r) + \delta_{\bar{n}}(T-r) + 1}} Y(u, z) v.$$

Again if $m = n$, then $u \circ_{g,n}^n v = u \circ_{g,n} v$ has been defined in [8]. Then, $O'_{g,n,n}(V) = O_{g,n}(V)$, $O_{g,0}(V) = O_g(V)$, $A_{g,n}(V) = V/O_{g,n}(V)$ and $A_{g,0}(V) = A_g(V)$ (cf. [7,8]).

Lemma 2.4. *For any weak g-twisted V-module M, $m, n \in (1/T)\mathbb{Z}_+$ and $a \in O'_{g,n,m}(V)$, we have $o_{m-n}(a) = 0$ on $\Omega_m(M)$.*

Proof. It is trivial if $a = L(-1)u + (L(0) + m - n)u$ for some $u \in V$. Assume that a has the form $u \circ_{g,m}^n v$ for some $u \in V^r$ and $v \in V$. Then for any $w \in \Omega_m(M)$, by the twisted Jacobi identity (2.1) we have

$$
\begin{aligned}
& o_{m-n}\left(u \circ_{g,m}^{n} v\right) w \\
&= \sum_{k \geq 0} \binom{\operatorname{wt} u+\lfloor m\rfloor+\delta_{\bar{n}}(T-r)+r/T-1}{k} o_{m-n}\left(u_{k-\lfloor m\rfloor-\lfloor n\rfloor-\delta_{\bar{m}}(r)-\delta_{\bar{n}}(T-r)-1} v\right) w \\
&= \sum_{k \geq 0}(-1)^{k}\binom{-\lfloor m\rfloor-\lfloor n\rfloor-\delta_{\bar{m}}(r)-\delta_{\bar{n}}(T-r)-1}{k} \times \\
& \quad \Big(u_{\operatorname{wt} u+r/T-\lfloor n\rfloor-\delta_{\bar{n}}(T-r)-2-k} v_{\operatorname{wt} v-1+m+k+1+\delta_{\bar{n}}(T-r)-r/T-\bar{n}/T}- \\
& \quad\quad (-1)^{\lfloor m\rfloor+\lfloor n\rfloor+\delta_{\bar{m}}(r)+\delta_{\bar{n}}(T-r)+1} \\
& \quad\quad \times v_{\operatorname{wt} v+\bar{m}/T-r/T-\delta_{\bar{m}}(r)-1-k} u_{\operatorname{wt} u-1+m+k+r/T+\delta_{\bar{m}}(r)-\bar{m}/T}\Big) w .
\end{aligned}
$$

But by the definition of $\Omega_m(M)$,

$$
v_{\operatorname{wt} v-1+m+k+1+\delta_{\bar{n}}(T-r)-r/T-\bar{n}/T} w=u_{\operatorname{wt} u-1+m+k+r/T+\delta_{\bar{m}}(r)-\bar{m}/T} w=0 \quad \text{for all } k \in \mathbb{Z}_{+}.
$$

Thus, $o_{m-n}\left(u \circ_{g,m}^{n} v\right)=0$ on $\Omega_m(M)$, completing the proof. □

The following theorem is from [8, Theorem 2.4 and Theorem 3.3].

Theorem 2.5. (1) *The product $\bar{*}_{g,n}^{n}=*_{g,n}=*_{g,n}^{n}$ induces the structure of an associative algebra on $A_{g,n}(V)$ with identity $\mathbf{1}+O_{g,n}(V)$.*

(2) *Suppose that M is a weak g-twisted V-module. Then there is a representation of the associative algebra $A_{g,n}(V)$ on $\Omega_n(M)$ induced by the map $a \mapsto o(a)=a_{\operatorname{wt} a-1}$ for $a \in V$. Moreover, if $M=\bigoplus_{k \in(1/T)\mathbb{Z}_{+}} M(k)$ is an admissible g-twisted V-module, then $\bigoplus_{0 \leq k \leq n} M(k) \subseteq \Omega_n(M)$ and for each $k \in(1/T)\mathbb{Z}$ such that $0 \leq k \leq n$, $M(k)$ is an $A_{g,n}(V)$-module.*

For any $a, b, c, u \in V$ and any $p_1, p_2, p_3 \in(1/T)\mathbb{Z}_{+}$, let $O_{g,n,m}^{\prime\prime}(V)$ be the linear span of

$$
u *_{g,m,p_3}^{n}\left(\left(a *_{g,p_1,p_2}^{p_3} b\right) *_{g,m,p_1}^{p_3} c-a *_{g,m,p_2}^{p_3}\left(b *_{g,m,p_1}^{p_2} c\right)\right).
$$

In particular, by (2.4) we have

$$
\left(a *_{g,p_1,p_2}^{n} b\right) *_{g,m,p_1}^{n} c-a *_{g,m,p_2}^{n}\left(b *_{g,m,p_1}^{p_2} c\right) \in O_{g,n,m}^{\prime\prime}(V). \tag{2.5}
$$

Let

$$
O_{g,n,m}^{\prime\prime\prime}(V)=\sum_{p_1, p_2 \in(1/T)\mathbb{Z}_{+}}\left(V *_{g,p_1,p_2}^{n} O_{g,p_2,p_1}^{\prime}(V)\right) *_{g,m,p_1}^{n} V
$$

and

$$O_{g,n,m}(V) = O'_{g,n,m}(V) + O''_{g,n,m}(V) + O'''_{g,n,m}(V).$$

Set

$$A_{g,n,m}(V) = V/O_{g,n,m}(V).$$

Theorem 2.6. [4] *Let V be a vertex operator algebra and $m, n \in (1/T)\mathbb{Z}_+$. Then $A_{g,n,m}(V)$ is an $A_{g,n}(V)$–$A_{g,m}(V)$-bimodule such that the left and right actions of $A_{g,n}(V)$ and $A_{g,m}(V)$ are induced by $\bar{*}^n_{g,m}$ and $*^n_{g,m}$, respectively.*

Let U be an $A_{g,m}(V)$-module. Set

$$M(U) = \bigoplus_{n \in (1/T)\mathbb{Z}_+} A_{g,n,m}(V) \otimes_{A_{g,m}(V)} U.$$

Then, $M(U)$ is $(1/T)\mathbb{Z}_+$-graded such that $M(U)(n) = A_{g,n,m}(V) \otimes_{A_{g,m}(V)} U$. For $u \in V^r, p \in r/T + \mathbb{Z}$ and $n \in (1/T)\mathbb{Z}$, define an operator u_p from $M(U)(n)$ to $M(U)(n + \mathrm{wt}\, u - p - 1)$ (with the convention that $M(U)(i) = 0$ if $i < 0$) by

$$\begin{aligned} & u_p\big((v + O_{g,n,m}(V) \otimes w\big) \\ &= \begin{cases} \big(u *^{\mathrm{wt}\, u - p - 1 + n}_{g,m,n} v + O_{g,\mathrm{wt}\, u - p - 1 + n, m}(V)\big) \otimes w, & \text{if } \mathrm{wt}\, u - 1 - p + n \geq 0, \\ \quad 0, & \text{if } \mathrm{wt}\, u - 1 - p + n < 0, \end{cases} \end{aligned}$$

for $v \in V$ and $w \in U$.

Theorem 2.7. [4] *Let U be an $A_{g,m}(V)$-module. Then*

$$M(U) = \bigoplus_{n \in (1/T)\mathbb{Z}_+} A_{g,n,m}(V) \otimes_{A_{g,m}(V)} U$$

is an admissible g-twisted V-module with $M(U)(n) = A_{g,n,m}(V) \otimes_{A_{g,m}(V)} U$ and has the following universal property: for any weak g-twisted V-module W and any $A_{g,m}(V)$-homomorphism $\phi : U \to \Omega_m(W)$, there is a unique homomorphism $\bar{\phi} : M(U) \to W$ of weak g-twisted V-modules which extends ϕ. Moreover, if U cannot factor through $A_{g,m-1/T}(V)$, then $M(U)(0) \neq 0$.

3. Associative algebras $\mathcal{A}_{g,n}(V)$ and $\mathcal{A}_{g,n}(V)$–$\mathcal{A}_{g,m}(V)$-bimodules $\mathcal{A}_{g,n,m}(V)$

In this section, we will construct a family of associative algebras $\mathcal{A}_{g,n}(V)$ and a family of $\mathcal{A}_{g,n}(V)$–$\mathcal{A}_{g,m}(V)$-bimodules $\mathcal{A}_{g,n,m}(V)$ from the perspective of representations, and show that they share the similar properties as for $A_{g,m}(V)$ and $A_{g,n,m}(V)$.

For any $m, n \in (1/T)\mathbb{Z}_+$, let

$$\mathcal{O}_{g,n,m}(V) = \left\{u \in V \mid o_{m-n}(u)|_{\Omega_m(M)} = 0 \text{ for any weak } g\text{-twisted } V\text{-module } M\right\}.$$

Set

$$\mathcal{A}_{g,n,m}(V) = V/\mathcal{O}_{g,n,m}(V).$$

Write $\mathcal{O}_{g,n,m}(V)$ as $\mathcal{O}_{g,n}(V)$ and $\mathcal{A}_{g,n,m}(V)$ as $\mathcal{A}_{g,n}(V)$ if $m = n$. And when $g = 1$, these $\mathcal{A}_{1,n,m}(V)$ were studied in [15].

The following lemma is clear from the definition of $\mathcal{O}_{g,n,m}(V)$.

Lemma 3.1. *$\mathcal{O}_{g,n,m}(V) \subseteq \mathcal{O}_{g,n-l,m-l}(V)$ for $(1/T)\mathbb{Z}_+ \ni l \leq \min\{m,n\}$.*

We have the following result from [4, Lemma 5.1].

Lemma 3.2. *Let M be a weak g-twisted V-module. Then*

$$o_{m-n}\left(u *_{g,m,p}^{n} v\right) = o_{p-n}(u) o_{m-p}(v) \text{ on } \Omega_m(M) \text{ for } u, v \in V \text{ and } m, n, p \in (1/T)\mathbb{Z}_+.$$

In particular, $u *_{g,m}^{n} \mathbf{1} - u \in \mathcal{O}_{g,n,m}(V)$.

By the definition of $\mathcal{O}_{g,n,m}(V)$, Lemma 2.3 and Lemma 3.2, it is not difficult to show the following statements.

Lemma 3.3. *Let $m, n \in (1/T)\mathbb{Z}_+$. Then*

(1) $\left(a *_{g,p_1,p_2}^{n} b\right) *_{g,m,p_1}^{n} c - a *_{g,m,p_2}^{n} \left(b *_{g,m,p_1}^{p_2} c\right) \in \mathcal{O}_{g,n,m}(V)$ for $a, b, c \in V$ and $p_1, p_2 \in (1/T)\mathbb{Z}_+$. In particular, $\left(a \bar{*}_{g,m}^{n} b\right) *_{g,m}^{n} c \equiv a \bar{*}_{g,m}^{n} \left(b *_{g,m}^{n} c\right) \mod \mathcal{O}_{g,n,m}(V)$.

(2) $V *_{g,m,p}^{n} \mathcal{O}_{g,p,m}(V) \subseteq \mathcal{O}_{g,n,m}(V)$ and $\mathcal{O}_{g,n,p}(V) *_{g,m,p}^{n} V \subseteq \mathcal{O}_{g,n,m}(V)$ for $p \in (1/T)\mathbb{Z}_+$. In particular, $V \bar{*}_{g,m}^{n} \mathcal{O}_{g,n,m}(V) \subseteq \mathcal{O}_{g,n,m}(V)$ and $\mathcal{O}_{g,n,m}(V) *_{g,m}^{n} V \subseteq \mathcal{O}_{g,n,m}(V)$; $\mathcal{O}_{g,n}(V) \bar{*}_{g,m}^{n} V \subseteq \mathcal{O}_{g,n,m}(V)$ and $V *_{g,m}^{n} \mathcal{O}_{g,m}(V) \subseteq \mathcal{O}_{g,n,m}(V)$.

Remark 3.4. (1) It is clear by Lemma 2.4, Lemma 3.2 and Lemma 3.3 that $O_{g,n,m}(V) = O'_{g,n,m}(V) + O''_{g,n,m}(V) + O'''_{g,n,m}(V) \subseteq \mathcal{O}_{g,n,m}(V)$. In particular, $O_{g,n}(V) \subseteq \mathcal{O}_{g,n}(V)$.

(2) Since $\mathcal{A}_{g,n}(V)$ is a quotient algebra of $A_{g,n}(V)$, every $\mathcal{A}_{g,n}(V)$-module automatically becomes an $A_{g,n}(V)$-module.

Theorem 3.5. *Let $m, n \in (1/T)\mathbb{Z}_+$. Then*

1. *$\mathcal{A}_{g,n}(V)$ is an associative algebra under the multiplication $\bar{*}_{g,n}^{n} = *_{g,n}^{n}$ with the identity $\mathbf{1} + \mathcal{O}_{g,n}(V)$.*
2. *$\mathcal{A}_{g,n,m}(V)$ is an $\mathcal{A}_{g,n}(V)$–$\mathcal{A}_{g,m}(V)$-bimodule with $\bar{*}_{g,m}^{n}$ the left action and $*_{g,m}^{n}$ the right action.*
3. *Suppose that M is a weak g-twisted V-module. Then there is a representation of $\mathcal{A}_{g,n}(V)$ on $\Omega_n(M)$ induced by the linear map $u \mapsto o(u)$ for $u \in V$. Moreover, if $M = \bigoplus_{k \in (1/T)\mathbb{Z}_+} M(k)$ is an admissible g-twisted V-module, then*

$$\bigoplus_{0\le k\le n} M(k) \subseteq \Omega_n(M)$$

and each $M(k)$ is an $\mathcal{A}_{g,n}(V)$-module for $0 \le k \le n$.

Proof. (1) By Lemma 3.3 (2), $\mathcal{O}_{g,n}(V)$ is a two-sided ideal of V under the multiplication $\bar{*}^n_{g,n} = *^n_{g,n}$. It follows from Lemma 3.3 (1) that this multiplication satisfies the associativity. Thus, $\mathcal{A}_{g,n}(V)$ is an associative algebra. And by (2.4) and Lemma 3.2 we see that $\mathbf{1} + \mathcal{O}_{g,n}(V)$ is its identity.

(2) Note that the left action of $\mathcal{A}_{g,n}(V)$ and the right action of $\mathcal{A}_{g,m}(V)$ on $\mathcal{A}_{g,n,m}(V)$ are well defined by Lemma 3.3 (2); and also that the two actions are compatible by Lemma 3.3 (1). Thus, $\mathcal{A}_{g,n,m}(V)$ is an $\mathcal{A}_{g,n}(V)$–$\mathcal{A}_{g,m}(V)$-bimodule.

(3) It is clear from the definition of $\mathcal{O}_{g,n}(V)$ that the given representation is well defined. And this map is an algebra homomorphism by Lemma 3.2, proving the first statement. The second statement follows from that $v_{\mathrm{wt}\, v-1+i}M(k) = 0$ for any $i > n$ and $k \le n$ and that $o(v)M(k) \subseteq M(k)$ for any $k \in (1/T)\mathbb{Z}_+$. $\square$

The following corollary is an immediate consequence of Lemma 3.1 and Theorem 3.5 (2).

Corollary 3.6. *For any $l, m, n \in (1/T)\mathbb{Z}_+$ such that $l \le \min\{m, n\}$, the identity map on V induces an epimorphism of $\mathcal{A}_{g,n}(V)$–$\mathcal{A}_{g,m}(V)$-bimodules from $\mathcal{A}_{g,n,m}(V)$ to $\mathcal{A}_{g,n-l,m-l}(V)$. In particular, the identity map on V induces an epimorphism of algebras from $\mathcal{A}_{g,n}(V)$ to $\mathcal{A}_{g,n-l}(V)$.*

Let U be an $\mathcal{A}_{g,m}(V)$-module. Set

$$\mathcal{M}(U) = \bigoplus_{n\in(1/T)\mathbb{Z}_+} \mathcal{A}_{g,n,m}(V) \otimes_{\mathcal{A}_{g,m}(V)} U.$$

Then $\mathcal{M}(U) = \bigoplus_{n\in(1/T)\mathbb{Z}_+} \mathcal{M}(U)(n)$ is $(1/T)\mathbb{Z}_+$-graded with $\mathcal{M}(U)(n) = \mathcal{A}_{g,n,m}(V) \otimes_{\mathcal{A}_{g,m}(V)} U$ for $n \in (1/T)\mathbb{Z}_+$. Following the construction of $M(U)$ (see Section 2), for $u \in V^r$ define the vertex operator $Y_{\mathcal{M}(U)}(u, z) = \sum_{p\in r/T+\mathbb{Z}} u_p z^{-p-1}$ with u_p being a linear map from $\mathcal{M}(U)(n)$ to $\mathcal{M}(U)(n + \mathrm{wt}\, u - p - 1)$ for $n \in (1/T)\mathbb{Z}_+$ (decreeing $\mathcal{M}(U)(k) = 0$ if $k < 0$) given by

$$u_p\big((v + \mathcal{O}_{g,n,m}(V)) \otimes w\big) = \big(u *^{n+\mathrm{wt}\, u-p-1}_{g,m,n} v + \mathcal{O}_{g,m,n+\mathrm{wt}\, u-p-1}(V)\big) \otimes w$$

for $v \in V$ and $w \in U$. This action is well defined, since for any $v \in \mathcal{A}_{g,n,m}(V), a \in \mathcal{A}_{g,m}(V)$ and $w \in U$, we have $u *^{n+\mathrm{wt}\, u-p-1}_{g,m,n} \mathcal{O}_{g,n,m}(V) \subseteq \mathcal{O}_{g,n+\mathrm{wt}\, u-p-1,m}(V)$ by Lemma 3.3 (2), and

$$
\begin{aligned}
u_p\left((v *_{g,m}^n a) \otimes w\right) &= \left(u *_{g,m,n}^{n+\mathrm{wt}\,u-p-1} \left(v *_{g,m}^n a\right)\right) \otimes w \\
&= \left((u *_{g,m,n}^{n+\mathrm{wt}\,u-p-1} v) *_{g,m}^{n+\mathrm{wt}\,u-p-1} a\right) \otimes w \\
&= \left(u *_{g,m,n}^{n+\mathrm{wt}\,u-p-1} v\right) \otimes a \cdot w = u_p(v \otimes a \cdot w)
\end{aligned}
$$

by Lemma 3.3 (1). Note that U can also be viewed as an $A_{g,m}(V)$-module by Remark 3.4 (2). Then we can define the linear map

$$
\psi_{n,m} : A_{g,n,m}(V) \otimes_{A_{g,m}(V)} U \longrightarrow \mathcal{A}_{g,n,m}(V) \otimes_{\mathcal{A}_{g,m}(V)} U
$$

sending $(u + O_{g,n,m}(V)) \otimes w$ to $(u + \mathcal{O}_{g,n,m}(V)) \otimes w$ for $u \in V$ and $w \in U$, which is well defined. Note that these $\psi_{n,m}$ for $n \in (1/T)\mathbb{Z}_+$ induce the surjective linear map $\psi : M(U) \longrightarrow \mathcal{M}(U)$ such that $\psi(u_p w) = u_p \psi(w)$ for any $p \in (1/T)\mathbb{Z}, u \in V$ and $w \in M(U)$. Thus, $\mathcal{M}(U)$ is a weak g-twisted V-module. Moreover, by definition of the action of u_p, $\mathcal{M}(U)$ is an admissible g-twisted V-module. Similarly, one can show that $\mathcal{M}(U)$ shares the same universal property as $M(U)$. Then we arrive at the following result.

Theorem 3.7. *Let U be an $\mathcal{A}_{g,m}(V)$-module. Then*

$$
\mathcal{M}(U) = \bigoplus_{n \in (1/T)\mathbb{Z}_+} \mathcal{A}_{g,n,m}(V) \otimes_{\mathcal{A}_{g,m}(V)} U
$$

is an admissible g-twisted V-module with $\mathcal{M}(U)(n) = \mathcal{A}_{g,n,m}(V) \otimes_{\mathcal{A}_{g,m}(V)} U$ satisfying the following universal property: for any weak g-twisted V-module W and any $\mathcal{A}_{g,m}(V)$-morphism $\phi : U \to \Omega_m(W)$, there is a unique homomorphism $\bar{\phi} : \mathcal{M}(U) \to W$ of weak g-twisted V-modules which extends ϕ. Moreover, $\mathcal{M}(U)(0) \neq 0$ if U cannot factor through $\mathcal{A}_{g,m-1/T}(V)$.

4. The universal enveloping algebra $U(V[g])$ of V with respect to g

In this section, we shall first recall the construction of the universal enveloping algebra $U(V[g])$ of V with respect to g and then use $U(V[g])$ to construct admissible g-twisted V-modules with some universal property.

Recall from [7] (see also [1]) the Lie algebra

$$
\hat{V}[g] = \mathcal{L}(V, g)/D\mathcal{L}(V, g),
$$

where $\mathcal{L}(V, g) = \bigoplus_{r=0}^{T-1} V^r \otimes \mathbb{C}t^{\frac{r}{T}}\left[t, t^{-1}\right]$ and $D = L(-1) \otimes \mathrm{id} + \mathrm{id} \otimes \frac{d}{dt}$. Denote by $u(m)$ the image of $u \otimes t^m$ in $\hat{V}[g]$. Then the Lie bracket on $\hat{V}[g]$ is given by

$$
\left[u(m + \frac{r}{T}), v(n + \frac{s}{T})\right] = \sum_{i=0}^{\infty} \binom{m + \frac{r}{T}}{i} (u_i v)(m + n + \frac{r+s}{T} - i)
$$

for $u \in V^r, v \in V^s$ and $m, n \in \mathbb{Z}$. If we define the degree of $u(m)$ to be $\mathrm{wt}\, u - m - 1$, then $\hat{V}[g]$ is a $(1/T)\mathbb{Z}$-graded Lie algebra, i.e., $\hat{V}[g] = \bigoplus_{m \in (1/T)\mathbb{Z}} \hat{V}[g]_m$ and $\left[\hat{V}[g]_i, \hat{V}[g]_j\right] \subseteq \hat{V}[g]_{i+j}$ for any $i, j \in (1/T)\mathbb{Z}$.

Let $U(\hat{V}[g])$ be the universal enveloping algebra of the Lie algebra $\hat{V}[g]$. Then the $(1/T)\mathbb{Z}$-grading on $\hat{V}[g]$ induces a $(1/T)\mathbb{Z}$-grading on $U(\hat{V}[g]) = \bigoplus_{m \in (1/T)\mathbb{Z}} U(\hat{V}[g])_m$. Set

$$U(\hat{V}[g])_m^k = \sum_{i \leq k, i \in (1/T)\mathbb{Z}} U(\hat{V}[g])_{m-i} U(\hat{V}[g])_i$$

for $0 > k \in (1/T)\mathbb{Z}$ and $U(\hat{V}[g])_m^0 = U(\hat{V}[g])_m$. Then

$$U(\hat{V}[g])_m^k \subseteq U(\hat{V}[g])_m^{k+1/T}$$

and

$$\bigcap_{k \in -(1/T)\mathbb{Z}_+} U(\hat{V}[g])_m^k = 0, \qquad \bigcup_{k \in -(1/T)\mathbb{Z}_+} U(\hat{V}[g])_m^k = U(\hat{V}[g])_m.$$

Thus, $\left\{U(\hat{V}[g])_m^k \mid k \in -(1/T)\mathbb{Z}_+\right\}$ forms a fundamental neighborhood system of $U(\hat{V}[g])_m$. Let $\widetilde{U}(\hat{V}[g])_m$ be the completion of $U(\hat{V}[g])_m$, then

$$\widetilde{U}(\hat{V}[g]) := \bigoplus_{m \in (1/T)\mathbb{Z}} \widetilde{U}(\hat{V}[g])_m$$

is a complete topological ring which allows infinite sums in it.

For each $m \in (1/T)\mathbb{Z}$, define a linear map $J_m(\cdot) : V \to \hat{V}[g]$ sending $u \in V^r$ to $u(\mathrm{wt}\, u + m - 1)$ if $m \in r/T + \mathbb{Z}$ and zero otherwise.

Definition 4.1. The universal enveloping algebra $U(V[g])$ of V with respect to g is the quotient of $\widetilde{U}(\hat{V}[g])$ by a two-sided ideal generated by the following relations:

$$\mathbf{1}(i) = \delta_{i,-1} \text{ for } i \in \mathbb{Z},$$
$$[\omega(i+1), \omega(j+1)] = (i-j)\omega(i+j+1) + \delta_{i+j,0}\frac{i^3 - i}{12}c \quad \text{for } i, j \in \mathbb{Z},$$

and

$$\sum_{i \geq 0} (-1)^i \binom{l}{i} \left(J_{s-i}(u) J_{t+i}(v) - (-1)^l J_{l+t-i}(v) J_{s+i-l}(u)\right)$$
$$= \sum_{i \geq 0} \binom{s + \mathrm{wt}\, u - l - 1}{i} J_{s+t}\left(u_{l+i} v\right) \quad \text{for } u \in V^r, v \in V^{r'}, l \in \mathbb{Z}, s \in \frac{r}{T} + \mathbb{Z}, t \in \frac{r'}{T} + \mathbb{Z}.$$

It is clear that $U(V[g]) = \bigoplus_{m\in(1/T)\mathbb{Z}} U(V[g])_m$ is a $(1/T)\mathbb{Z}$-graded associative algebra. Set

$$U(V[g])_m^k = \sum_{i\le k, i\in(1/T)\mathbb{Z}} U(V[g])_{m-i}U(V[g])_i$$

for $0 > k \in (1/T)\mathbb{Z}$. Then, $U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}$ is a $U(V[g])_0/U(V[g])_0^{-n-1/T}$–$U(V[g])_0/U(V[g])_0^{-m-1/T}$-bimodule.

Theorem 4.2. *[16] The linear map $\varphi : V \to U(V[g])_0$ sending u to $J_0(u)$ induces an algebra isomorphism φ_n between $A_{g,n}(V)$ and $U(V[g])_0/U(V[g])_0^{-n-1/T}$ for each $n \in (1/T)\mathbb{Z}_+$.*

Remark 4.3. (1) Suppose that M is a weak g-twisted V-module. Then by Theorem 4.2 and Theorem 2.5 (2), there is a representation of $U(V[g])_0/U(V[g])_0^{-n-1/T}$ on $\Omega_n(M)$ induced by the linear map $a \to o(a) = a_{\mathrm{wt}\, a-1}$ for $a \in V$.

(2) From the construction of $U(V[g])$, any weak g-twisted V-module is naturally a $U(V[g])$-module with the action induced by the map $u(m) \mapsto u_m$ for any $u \in V^r$ and $m \in r/T + \mathbb{Z}$.

Theorem 4.2 tells us that $A_{g,n}(V)$ can be realized as some quotient of $U(V[g])_0$. And in Section 5 we are going to make use of the subspace $U(V[g])_{n-m}$ to realize the bimodule $A_{g,n,m}(V)$.

Let U be a $U(V[g])_0/U(V[g])_0^{-m-1/T}$-module. Set

$$\mathbb{M}(U) = \bigoplus_{n\in(1/T)\mathbb{Z}_+} U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T} \otimes_{U(V[g])_0/U(V[g])_0^{-m-1/T}} U.$$

Set $\mathbb{M}(U) = \bigoplus_{n\in(1/T)\mathbb{Z}_+} \mathbb{M}(U)(n)$ with

$$\mathbb{M}(U)(n) = U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T} \otimes_{U(V[g])_0/U(V[g])_0^{-m-1/T}} U$$

for $n \in (1/T)\mathbb{Z}_+$. Then, $\mathbb{M}(U)(m) \cong U$ as $A_{g,m}(V)$-modules by Theorem 4.2.

Also following the construction of $M(U)$, we equip $\mathbb{M}(U)$ with the vertex operator maps $Y_{\mathbb{M}(U)}(u, z) = \sum_{p\in r/T+\mathbb{Z}} u_p z^{-p-1}$ for $u \in V^r$, where for $n \in (1/T)\mathbb{Z}_+$, the linear map u_p from $\mathbb{M}(U)(n)$ to $\mathbb{M}(U)(n + \mathrm{wt}\, u - p - 1)$ is defined as follows:

$$u_p(v \otimes w) = \begin{cases} u(p)v \otimes w, & \text{if } n + \mathrm{wt}\, u - p - 1 \ge 0, \\ 0, & \text{if } n + \mathrm{wt}\, u - p - 1 < 0, \end{cases}$$

for $v \in U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}$ and $w \in U$. Then $\mathbb{M}(U)$ is an admissible g-twisted V-module, since the twisted Jacobi identity follows immediately from the construction of $U(V[g])$; and $\mathbb{M}(U)$ is generated by $\mathbb{M}(U)(m)$.

Theorem 4.4. *Let U be a $U(V[g])_0/U(V[g])_0^{-m-1/T}$-module. Then the admissible g-twisted V-module*

$$\mathbb{M}(U) = \bigoplus_{n\in(1/T)\mathbb{Z}_+} U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T} \otimes_{U(V[g])_0/U(V[g])_0^{-m-1/T}} U$$

has the following universal property: for any weak g-twisted V-module W and any $A_{g,m}(V)$-morphism $\phi : U \to \Omega_m(W)$, there is a unique homomorphism $\bar{\phi} : \mathbb{M}(U) \to W$ of weak g-twisted V-modules which extends ϕ. Moreover, if U cannot factor through $A_{g,m-1/T}(V)$, then $\mathbb{M}(U)(0) \neq 0$.

Proof. Define $\bar{\phi} : \mathbb{M}(U) \to W$ by $\bar{\phi}(u\otimes w) = u\phi(w)$ for $u \in U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}$ and $w \in U$. Note that the action of $U(V[g])_{n-m}^{-m-1/T}$ on $\Omega_m(W)$ is trivial by Remark 4.3 (2) and also that

$$\bar{\phi}(u\cdot v\otimes w) = (u\cdot v)\phi(w) = u\phi(v\cdot w) = \bar{\phi}(u\otimes v\cdot w) \quad \text{for } v \in U(V[g])_0/U(V[g])_0^{-m-1/T}.$$

Thus, $\bar{\phi}$ is well defined. It is clear that $\bar{\phi}|_U = \phi$ by regarding

$$U = \mathbb{M}(U)(m) = U(V[g])_0/U(V[g])_0^{-m-1/T} \otimes_{U(V[g])_0/U(V[g])_0^{-m-1/T}} U.$$

And for $v \in V$, again by Remark 4.3 (2), we have

$$\begin{aligned}\bar{\phi}\,(v_p(u\otimes w)) &= \bar{\phi}(v(p)u\otimes v) = (v(p)u)\phi(w) = v(p)(u\phi(w)) = v(p)(\bar{\phi}(u\otimes w))\\ &= v_p(\bar{\phi}(u\otimes w)).\end{aligned}$$

Thus, $\bar{\phi}$ is a homomorphism of weak g-twisted V-modules, whose uniqueness follows from the fact that $\mathbb{M}(U)$ is generated by $U = \mathbb{M}(U)(m)$. $\square$

5. Isomorphisms

In this section we shall show $O_{g,n,m}(V) = \mathcal{O}_{g,n,m}(V)$ and realize $A_{g,n,m}(V)$ as some quotient of $U(V[g])_{n-m}$.

Following from [4, Lemma 5.1], we have:

Lemma 5.1. *Let $u, v \in V$ and $m, n, p \in (1/T)\mathbb{Z}_+$. Then*

$$J_{m-n}\left(u *_{g,m,p}^{n} v\right) \equiv J_{p-n}(u)J_{m-p}(v) \mod U(V[g])_{n-m}^{-m-1/T}.$$

The main result of this section is as follows.

Theorem 5.2. (1) *$A_{g,m}(V) = \mathcal{A}_{g,m}(V) = A_{g,m,m}(V)$ and $\mathcal{A}_{g,n,m}(V) = A_{g,n,m}(V)$ for any $m, n \in (1/T)\mathbb{Z}_+$.*

(2) *The* $A_{g,n}(V)$–$A_{g,m}(V)$*-bimodules* $A_{g,n,m}(V)$ *and* $U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}$ *are isomorphic for any* $m,n\in(1/T)\mathbb{Z}_+$.

Proof. Fix $m\in(1/T)\mathbb{Z}_+$ and take $U=A_{g,m}(V)$. By Theorem 4.2,

$$A_{g,m}(V)\cong\mathbb{M}(U)(m)=U(V[g])_0/U(V[g])_0^{-m-1/T}\otimes_{U(V[g])_0/U(V[g])_0^{-m-1/T}}A_{g,m}(V).$$

Now it follows from Theorem 3.5 (3) that the multiplication $\bar{*}_{g,m}^m=*_{g,m}^m$ on V induces an $\mathcal{A}_{g,m}(V)$-module structure on $A_{g,m}(V)$. In particular, we have $\mathcal{O}_{g,m}(V)\subseteq O_{g,m}(V)$, which together with Remark 3.4 (1) gives $\mathcal{O}_{g,m}(V)=O_{g,m}(V)$, i.e., $\mathcal{A}_{g,m}(V)=A_{g,m}(V)$. Similarly, when replacing $\mathbb{M}(U)$ by $M(U)$ we can obtain $A_{g,m,m}(V)=\mathcal{A}_{g,m}(V)$. That is, both $M(U)$ and $\mathcal{M}(U)$ have the same generating set. Then $M(U)=\mathcal{M}(U)$, since these two modules have the same universal property by Theorems 2.7 and 3.7. Now by considering $M(U)(n)$ and $\mathcal{M}(U)(n)$ we see that $A_{g,n,m}(V)=\mathcal{A}_{g,n,m}(V)$, proving (1).

For convenience, we would identify $A_{g,n,m}(V)\otimes_{A_{g,m}(V)}A_{g,m}(V)$ with $A_{g,n,m}(V)$, and identify $U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}\otimes_{U(V[g])_0/U(V[g])_0^{-m-1/T}}A_{g,m}(V)$ with $U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}$. It follows from (1) and Theorem 4.2 that the linear map

$$\begin{aligned}\varphi_{m,m}:A_{g,m,m}(V)&\to U(V[g])_0/U(V[g])_0^{-m-1/T}\\ u+O_{g,m,m}(V)&\mapsto J_0(u)+U(V[g])_0^{-m-1/T}\end{aligned}$$

is an isomorphism of $A_{g,m}(V)$-modules. Now by Theorem 2.7 and Theorem 4.4, this map can be extended to an isomorphism of admissible g-twisted V-modules from $M(U)$ to $\mathbb{M}(U)$ such that for any $n\in(1/T)\mathbb{Z}_+$, the linear map

$$\begin{aligned}\varphi_{n,m}:A_{g,n,m}(V)&\to U(V[g])_{n-m}/U(V[g])_{n-m}^{-m-1/T}\\ u+O_{g,n,m}(V)&\mapsto J_{m-n}(u)+U(V[g])_{n-m}^{-m-1/T}\end{aligned}$$

gives an isomorphism of $A_{g,n}(V)$-modules. In fact, this is also a homomorphism of right $A_{g,m}(V)$-modules:

$$\begin{aligned}&\varphi_{n,m}\left((u+O_{g,n,m}(V))*_{g,m}^m(b+O_{g,m}(V))\right)=\varphi_{n,m}\left(u*_{g,m}^n b+O_{g,n,m}(V)\right)\\ =&J_{m-n}\left(u*_{g,m}^n b\right)+U(V[g])_{n-m}^{-m-1/T}=J_{m-n}(u)J_0(b)+U(V[g])_{n-m}^{-m-1/T}\\ =&\left(J_{m-n}(u)+U(V[g])_{n-m}^{-m-1/T}\right)\cdot(b+O_{g,m}(V))\\ =&\varphi_{n,m}\left(u+O_{g,n,m}(V)\right)\cdot\left(b+O_{g,m}(V)\right),\end{aligned}$$

where the third equality follows from Lemma 5.1. Thus $\varphi_{n,m}$ is an $A_{g,n}(V)$–$A_{g,m}(V)$-bimodule isomorphism, proving (2). □

By Theorem 5.2 and Corollary 3.6, one can obtain the following result.

Corollary 5.3. (1) *The identity map on V induces an epimorphism of $A_{g,n}(V)$–$A_{g,m}(V)$-bimodules from $A_{g,n,m}(V)$ to $A_{g,n-l,m-l}(V)$ for $l \in (1/T)\mathbb{Z}_+$ such that $l \leq \min\{m,n\}$.*

(2) *$O_{g,n,n}(V) = O_{g,n}(V)$ and*

$$\begin{aligned} O_{g,n,m}(V) &= \left\{ u \in V \mid J_{m-n}(u) \in U(V[g])_{n-m}^{-m-1/T} \right\} \\ &= \left\{ u \in V \,|o_{m-n}(u)|_{\Omega_m(M)} = 0 \text{ for any weak } g\text{-twisted } V\text{-module } M \right\} \end{aligned}$$

Remark 5.4. The equality $A_{g,n,n}(V) = A_{g,n}(V)$ and Corollary 5.3 (1) were first proved in [4].

6. Refining bimodules

In this section we shall refine the definition of the $A_{g,n}(V)$–$A_{g,m}(V)$-bimodule $A_{g,n,m}(V)$.

Notation 6.1. Until further notice we shall use the following conventions.

(1) For $m \in (1/T)\mathbb{Z}$ and $i \in \mathbb{Z}$, $\binom{m}{i} = 1$ if $i = 0$, and 0 if $i < 0$.

(2) For $k, l \in (1/T)\mathbb{Z}$, define $\sum\limits_{i=k}^{l} a_i = \sum_{i \in \mathbb{Z}_{k,l}} a_i$, where $\mathbb{Z}_{k,l} = \begin{cases} \mathbb{Z} \cap [l,k] & \text{if } l \leq k, \\ \mathbb{Z} \cap [k,l] & \text{if } l > k. \end{cases}$

(3) For $n \in (1/T)\mathbb{Z}_+$, $a \in V^r$ and $b \in V$, denote $f_i(a,b)$ as follows:

$$f_i(a,b) = \frac{(1+z)^{\mathrm{wt}\, a + q}}{z^i} Y(a,z)b \quad \text{for } i \in \mathbb{Z},$$

where $q = -1 + \lfloor n \rfloor + \delta_{\bar{n}}(r) + r/T$.

Lemma 6.2. *Let $n \in (1/T)\mathbb{Z}$ and $l \in \mathbb{Z}$. Then*

$$\sum_{j=0}^{n+1+l} (-1)^j \binom{l}{j} \sum_{i=0}^{n+1+l-j} (-1)^i \binom{-l+i+j-1}{i} \frac{1}{z^{i+j}} = 1.$$

Proof. This formula follows from [15, Lemma 3.8] if $n+1+l \geq 0$ and Notation 6.1 (2) if $n+1+l < 0$. □

Lemma 6.3. *For $k, n \in (1/T)\mathbb{Z}_+$, $a \in V^r, b \in V$ and $j, l \in \mathbb{Z}$, we have*

$$a *_{g,n,k+1+q+l-j}^{k} b = \sum_{i=0}^{k+1+q+l-j} (-1)^i \binom{-l+i+j-1}{i} \mathrm{Res}_z\, f_{i+j-l}(a,b),$$

where $q = -1 + \lfloor n \rfloor + \delta_{\bar{n}}(r) + r/T$.

Proof. Note that $\lfloor k+1+q+l-j \rfloor = \lfloor n \rfloor + \lfloor k+r/T \rfloor + \delta_{\bar{n}}(r) + l - j$ and $\lfloor k \rfloor - \lfloor k + r/T \rfloor + \delta_{\bar{k}}(T-r) = 0$. Then this lemma follows from the definition of product $*^{n}_{g,m,p}$ and Notation 6.1 (2)-(3). □

Set

$$M_g^{(m)} = \bigoplus_{n\in(1/T)\mathbb{Z}_+} V/O''_{g,n,m}(V),$$

which is clearly $(1/T)\mathbb{Z}_+$-graded with $M_g^{(m)}(n) = V/O''_{g,n,m}(V)$. For $u \in V$ and $p \in (1/T)\mathbb{Z}$, define the vertex operator map

$$u_p(v+O''_{g,n,m}(V)) = \begin{cases} u *^{n+\mathrm{wt}\,u-p-1}_{g,m,n} v + O''_{g,n+\mathrm{wt}\,u-p-1,m}(V), & \text{if } n+\mathrm{wt}\,u-p-1 \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

This action is well defined by the proof in [4, Lemma 3.8].

Lemma 6.4. *Let $m, n \in (1/T)\mathbb{Z}_+$. Then*

(1) *for any $u \in V^r$ and $p \in r/T + \mathbb{Z}$, $u_p(M_g^{(m)}(n)) = 0$ if $p > \mathrm{wt}\,u + n - 1$;*
(2) $Y_{M_g^{(m)}}(\mathbf{1}, z) = \mathrm{id}$*;*
(3) *for any $a \in V^r$ and $b \in V^s$, we have*

$$(z_2+z_0)^{\mathrm{wt}\,a+q}\, Y_{M_g^{(m)}}(Y(a,z_0)b, z_2) = (z_0+z_2)^{\mathrm{wt}\,a+q}\, Y_{M_g^{(m)}}(a, z_0+z_2)\, Y_{M_g^{(m)}}(b, z_2)$$

or equivalently, for any $l \in \mathbb{Z}$,

$$\begin{aligned} &\mathrm{Res}_{z_0} z_0^l (z_2+z_0)^{\mathrm{wt}\,a+q} z_2^{\mathrm{wt}\,b-q} Y_{M_g^{(m)}}(Y(a,z_0)b, z_2) \\ =&\mathrm{Res}_{z_0} z_0^l (z_0+z_2)^{\mathrm{wt}\,a+q} z_2^{\mathrm{wt}\,b-q} Y_{M_g^{(m)}}(a, z_0+z_2)\, Y_{M_g^{(m)}}(b, z_2) \end{aligned}$$

on $M_g^{(m)}(n)$, where $q = -1 + \lfloor n \rfloor + \delta_{\bar{n}}(r) + r/T$.

Proof. (1) follows immediately from the definition of u_p. And for (2), it is sufficient to show $\mathbf{1}_p = \delta_{p,-1}\,\mathrm{id}$ on $M_g^{(m)}(n)$ for any $n \in (1/T)\mathbb{Z}_+$. By (1), $\mathbf{1}_p = 0$ on $M_g^{(m)}(n)$ if $p > n-1$. Now considering $\mathbb{Z} \ni p \leq n-1$, then for any $v \in V$, we have

$$\begin{aligned} &\mathbf{1}_p(v + O''_{g,n,m}(V)) = \mathbf{1} *^{n-p-1}_{g,m,n} v + O''_{g,n-p-1,m}(V) \\ =&\sum_{i=0}^{\lfloor n \rfloor} (-1)^i \binom{\lfloor m \rfloor + \lfloor n-p-1 \rfloor - \lfloor n \rfloor + i}{i} \\ &\cdot \mathrm{Res}_z \frac{(1+z)^{\lfloor m \rfloor}}{z^{\lfloor m \rfloor + \lfloor n-p-1 \rfloor - \lfloor n \rfloor + i + 1}} Y(\mathbf{1}, z)v + O''_{g,n-p-1,m}(V) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\lfloor n\rfloor}(-1)^i\binom{\lfloor m\rfloor+\lfloor n-p-1\rfloor-\lfloor n\rfloor+i}{i}\\
&\qquad\qquad\cdot\binom{\lfloor m\rfloor}{\lfloor m\rfloor+\lfloor n-p-1\rfloor-\lfloor n\rfloor+i}v+O''_{g,n-p-1,m}(V)\\
&= \sum_{i=0}^{\lfloor n\rfloor}(-1)^i\binom{\lfloor m\rfloor-p+i-1}{i}\binom{\lfloor m\rfloor}{\lfloor m\rfloor-p+i-1}v+O''_{g,n-p-1,m}(V)\\
&= \sum_{i=0}^{\lfloor n\rfloor}(-1)^i\binom{\lfloor m\rfloor}{p+1}\binom{p+1}{i}v+O''_{g,n-p-1,m}(V)\\
&= \delta_{p,-1}v+O''_{g,n-p-1,m}(V) \qquad\qquad \text{(by Notation 6.1 (1))}\\
&= \delta_{p,-1}(v+O''_{g,n,m}(V)).
\end{aligned}
$$

Thus, (2) holds.

The idea of the proof of the third statement comes essentially from [4, Lemma 5.10] (see also [15, Lemma 3.10]). For $v+O''_{g,n,m}(V)\in M_g^{(m)}(n)$, $q=-1+\lfloor n\rfloor+\delta_{\bar{n}}(r)+r/T$ and let $\alpha\in\{0,\dots,T-1\}$ be such that $\alpha\equiv\bar{n}-r-s \mod T$, we have

$$
\begin{aligned}
&\operatorname{Res}_{z_0} z_0^l\,(z_2+z_0)^{\mathrm{wt}\,a+q}\,z_2^{\mathrm{wt}\,b-q}Y_{M_g^{(m)}}\left(Y\left(a,z_0\right)b,z_2\right)\left(v+O''_{g,n,m}(V)\right)\\
&= \sum_{j\in\mathbb{Z}_+}\binom{\mathrm{wt}\,a+q}{j}z_2^{\mathrm{wt}\,a+\mathrm{wt}\,b-j}Y_{M_g^{(m)}}\left(a_{j+l}b,z_2\right)\left(v+O''_{g,n,m}(V)\right)\\
&= \sum_{j\in\mathbb{Z}_+}\binom{\mathrm{wt}\,a+q}{j}\sum_{k\in\frac{\alpha}{T}+\mathbb{Z}_+}z_2^{l+k-n+1}\left(a_{j+l}b\right)_{\mathrm{wt}\,a+\mathrm{wt}\,b-j-l-2-k+n}\left(v+O''_{g,n,m}(V)\right)\\
&= \sum_{k\in\frac{\alpha}{T}+\mathbb{Z}_+}z_2^{l+k-n+1}\sum_{j\in\mathbb{Z}_+}\binom{\mathrm{wt}\,a+q}{j}\left(a_{j+l}b\right)*^k_{g,m,n}v+O''_{g,k,m}(V)\\
&= \sum_{k\in\frac{\alpha}{T}+\mathbb{Z}_+}z_2^{l+k-n+1}\Big(\operatorname{Res}_z\frac{(1+z)^{\mathrm{wt}\,a+q}}{z^{-l}}Y(a,z)b\Big)*^k_{g,m,n}v+O''_{g,k,m}(V)\\
&= \sum_{k\in\frac{\alpha}{T}+\mathbb{Z}_+}z_2^{l+k-n+1}\operatorname{Res}_z\left(f_{-l}(a,b)*^k_{g,m,n}v\right)+O''_{g,k,m}(V) \qquad \text{(by Notation 6.1(3))}\\
&= \sum_{k\in\frac{\alpha}{T}+\mathbb{Z}_+}z_2^{l+k-n+1}\sum_{j=0}^{k+1+q+l}(-1)^j\binom{l}{j}\sum_{i=0}^{k+1+q+l-j}(-1)^i\binom{-l+i+j-1}{i}\\
&\quad\times\operatorname{Res}_z\left(f_{i+j-l}(a,b)*^k_{g,m,n}v\right)+O''_{g,k,m}(V) \quad \text{(by Notation 6.1 (2)-(3) and Lemma 6.2)}\\
&= \sum_{k\in\frac{\alpha}{T}+\mathbb{Z}_+}z_2^{l+k-n+1}\sum_{j=0}^{k+1+q+l}(-1)^j\binom{l}{j}\left(\left(a*^k_{g,n,k+1+q+l-j}b\right)*^k_{g,m,n}v\right)+O''_{g,k,m}(V)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 6.3)}
\end{aligned}
$$

$$= \sum_{\substack{k \in \frac{\alpha}{T}+\mathbb{Z}_+ \\ k+1+l+q \geq 0}} z_2^{l+k-n+1} \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{l}{j} a *_{g,m,k+1+q+l-j}^{k} \left(b *_{g,m,n}^{k+1+q+l-j} v\right) + O''_{g,k,m}(V)$$

(by (2.5) and Notation 6.1 (1)-(2))

$$= \sum_{j \in \mathbb{Z}_+} \sum_{\substack{-n \leq i \in -s/T+\mathbb{Z} \\ -l+i+j \geq 1+q-n}} \binom{l}{j} (-1)^j z_2^{i+j-q} a *_{g,m,n+i}^{-l+i+j-1-q+n} \left(b *_{g,m,n}^{n+i} v\right)$$

$$+ O''_{g,-l+i+j-1-q+n,m}(V)$$

$$= \sum_{j \in \mathbb{Z}_+} \binom{l}{j} (-1)^j a_{\mathrm{wt}\, a+q+l-j} \sum_{-n \leq i \in -s/T+\mathbb{Z}} z_2^{i+j-q} b_{\mathrm{wt}\, b-1-i}(v + O''_{g,n,m}(V))$$

$$= \sum_{j \in \mathbb{Z}_+} \binom{l}{j} (-1)^j a_{\mathrm{wt}\, a+q+l-j} z_2^{\mathrm{wt}\, b+j-q} Y_{M_g^{(m)}}(b, z_2)(v + O''_{g,n,m}(V))$$

$$= \mathrm{Res}_{z_0} z_0^l (z_0+z_2)^{\mathrm{wt}\, a+q} z_2^{\mathrm{wt}\, b-q} Y_{M_g^{(m)}}(a, z_0+z_2) Y_{M_g^{(m)}}(b, z_2)(v + O''_{g,n,m}(V)),$$

proving (3). □

As an immediate consequence of Lemma 6.4 and Theorem 2.1 we have:

Proposition 6.5. *For any $m \in (1/T)\mathbb{Z}_+$, $M_g^{(m)}$ is an admissible g-twisted V-module.*

Recall from Section 2 that

$$L_{n,m}(V) = \mathrm{span}\left\{(L(-1) + L(0) + m - n)u \mid u \in V\right\}.$$

From [4, Lemma 3.1] and the definition of $O'_{g,n,m}(V)$, we know:

Lemma 6.6. *For any $m, n \in (1/T)\mathbb{Z}_+$,*

$$\bigoplus_{s \not\equiv \bar{m}-\bar{n} \bmod T} V^s + L_{n,m}(V) \subseteq O'_{g,n,m}(V).$$

For any $u, v \in V$, it follows from [28] (see also [4]) that

$$Y(v,z)u \equiv (1+z)^{-\mathrm{wt}\, u - \mathrm{wt}\, v - m + n} Y\left(u, \frac{-z}{1+z}\right) v \bmod L_{n,m}(V).$$

Then the following result, in fact, was proved already in [4, Lemma 3.4 and Corollary 3.5].

Lemma 6.7. *For $u \in V^r$ and $v \in V^s$, if $\bar{p} - \bar{n} \equiv r \bmod T, \bar{m} - \bar{p} \equiv s \bmod T$ and $m+n-p \geq 0$, then*

$$u *^n_{g,m,p} v - v *^n_{g,m,m+n-p} u - \mathrm{Res}_z (1+z)^{\mathrm{wt}u-1+p-n} Y(u,z)v \in L_{n,m}(V).$$

In particular, taking $p = m$ and $v = \mathbf{1}$ we have

$$u *^n_{g,m} \mathbf{1} - u \in L_{n,m}(V).$$

Theorem 6.8. *For any $m, n \in (1/T)\mathbb{Z}_+$,*

$$O_{g,n,m}(V) = \bigoplus_{s \not\equiv \bar{m}-\bar{n} \bmod T} V^s + L_{n,m}(V) + O''_{g,n,m}(V).$$

In particular, $O_{1,n,m}(V) = L_{n,m}(V) + O''_{1,n,m}(V)$.

Proof. By Proposition 6.5 and Theorem 3.5 (3), $V/O''_{g,m,m}(V) \subseteq \Omega_m(M_g^{(m)})$. Note by the definition of $O_{g,n,m}(V)$ that $O_{g,n,m}(V) = \bigoplus_{r=0}^{T-1}(O_{g,n,m}(V) \cap V^r)$. For any $u \in O_{g,n,m}(V) \cap V^r = \mathcal{O}_{g,n,m}(V) \cap V^r$ (see Theorem 5.2), then by the definition of $\mathcal{O}_{g,n,m}(V)$,

$$0 = o_{m-n}(u)(\mathbf{1} + O''_{g,m,m}(V)) = u *^n_{g,m} \mathbf{1} + O''_{g,n,m}(V),$$

i.e.,

$$u *^n_{g,m} \mathbf{1} \in O''_{g,n,m}(V).$$

If $\bar{m} - \bar{n} \equiv r \bmod T$, then by Lemma 6.7

$$u = u - u *^n_{g,m} \mathbf{1} + u *^n_{g,m} \mathbf{1} \in L_{n,m}(V) + O''_{g,n,m}(V);$$

otherwise, $u \in \bigoplus_{s \not\equiv \bar{m}-\bar{n} \bmod T} V^s$. Thus by Lemma 6.6,

$$O_{g,n,m}(V) = \bigoplus_{s \not\equiv \bar{m}-\bar{n} \bmod T} V^s + L_{n,m}(V) + O''_{g,n,m}(V).$$

And when $g = 1$, it is clear that $O_{1,n,m}(V) = L_{n,m}(V) + O''_{1,n,m}(V)$. $\square$

7. Appendix

A detailed proof of Theorem 2.1 is given in this appendix.

Proposition 7.1. *The twisted Jacobi identity (2.1) for $u \in V^r, v \in V^s$ and $w \in M$ is equivalent to: for any $w' \in M^* = \mathrm{Hom}(M, \mathbb{C})$, there exist $l \in \mathbb{Z}_+, f(z_1, z_2) \in \mathbb{C}((z_1, z_2))$ and $g(z_0, z_2) \in \mathbb{C}((z_0, z_2))$ such that*

$$\left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u, z_1) Y_M(v, z_2) w \right\rangle = f(z_1, z_2)(z_1 - z_2)^{-l}, \tag{D1}$$

$$\left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(v, z_2) Y_M(u, z_1) w \right\rangle = f(z_1, z_2)(-z_2 + z_1)^{-l}, \tag{D2}$$

$$\left\langle w', (z_2 + z_0)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(Y(u, z_0) v, z_2) w \right\rangle = g(z_0, z_2)(z_2 + z_0)^{-l}, \tag{D3}$$

$$\left\langle w', (z_0 + z_2)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u, z_0 + z_2) Y_M(v, z_2) w \right\rangle = g(z_0, z_2)(z_0 + z_2)^{-l}, \tag{D4}$$

$$f(z_2 + z_0, z_2) z_0^{-l} = g(z_0, z_2)(z_2 + z_0)^{-l}. \tag{D5}$$

Proof. Assume that there exist $l \in \mathbb{Z}_+$, $f(z_1, z_2)$ and $g(z_0, z_2)$ such that (D1)-(D5) hold. Recall from [5,13] that

$$z_1^{-1}\left(\frac{z_2 + z_0}{z_1}\right)^{\gamma} \delta\left(\frac{z_2 + z_0}{z_1}\right) = z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-\gamma} \delta\left(\frac{z_1 - z_0}{z_2}\right) \tag{7.1}$$

for $\gamma \in \mathbb{C}$. Thus, (2.1) is equivalent to: for any $w' \in M^*$,

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u, z_1) Y_M(v, z_2) w \right\rangle \\
&-z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)\left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(v, z_2) Y_M(u, z_1) w \right\rangle \\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)\left\langle w', (z_2 + z_0)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(Y(u, z_0) v, z_2) w \right\rangle,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) f(z_1, z_2)(z_1 - z_2)^{-l} - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) f(z_1, z_2)(-z_2 + z_1)^{-l} \\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) f(z_2 + z_0, z_2) z_0^{-l}
\end{aligned}$$

according to (D1)−(D3) and (D5). But this follows immediately from multiplying

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) = z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right)$$

by $f(z_1, z_2) z_0^{-l}$. Therefore, we get (2.1).

Conversely, assume that the commutator formula (2.3) and the weak associativity (2.2) hold. Moreover, we may choose l large enough such that $z^{l+\frac{r}{T}} Y_M(u, z) w$ involves only nonnegative integral powers of z. Then, it follows from (2.3) and (7.1) that

$$[Y_M(u, z_1), Y_M(v, z_2)] = \mathrm{Res}_{z_0} Y_M\left(z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(u, z_0) v, z_2\right)\left(\frac{z_1 - z_0}{z_2}\right)^{-\frac{r}{T}}$$

$$
\begin{aligned}
&= \operatorname{Reg}_{z_0} Y_M \left(\left(z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\right) Y(u,z_0)v, z_2\right)\left(\frac{z_1-z_0}{z_2}\right)^{-\frac{r}{T}} \\
&= \operatorname{Res}_{z_0} Y_M \left(z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y(u,z_1-z_2)v \right. \\
&\qquad\qquad \left. - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right) Y(u,-z_2+z_1)v, z_2\right)\left(\frac{z_1-z_0}{z_2}\right)^{-\frac{r}{T}} \\
&= z_2^{\frac{r}{T}} z_1^{-\frac{r}{T}} \sum_{n\ge 0} \operatorname{Res}_{z_0} Y_M \left(z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y(u,z_1-z_2)v \right. \\
&\qquad\qquad \left. - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right) Y(u,-z_2+z_1)v, z_2\right)\binom{-r/T}{n}\left(\frac{-z_0}{z_1}\right)^{n} \\
&= z_2^{\frac{r}{T}} z_1^{-\frac{r}{T}} \sum_{n=0}^{N} (-1)^n \binom{-r/T}{n} z_1^{-n} Y_M\Big(\big(Y(u,z_1-z_2)(z_1-z_2)^n \\
&\qquad\qquad - Y(u,-z_2+z_1)(-z_2+z_1)^n\big)v, z_2\Big),
\end{aligned}
$$

where N is a nonnegative integer such that $z^{N+1}Y(u,z)v \in V[[z]]$. Thus, there exists $l \in \mathbb{Z}_+$ such that

$$(z_1-z_2)^l \left[Y_M(u,z_1), Y_M(v,z_2)\right] = 0.$$

Then for any $w \in M$,

$$
\begin{aligned}
&(z_1-z_2)^l \left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u,z_1) Y_M(v,z_2) w\right\rangle \\
&= (-z_2+z_1)^l \left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(v,z_2) Y_M(u,z_1) w\right\rangle,
\end{aligned}
$$

which is denoted by $f(z_1,z_2)$. Note that both sides of the above formula involve only finitely many negative powers of z_2 and z_1. Thus, $f(z_1,z_2) \in \mathbb{C}((z_1,z_2))$, proving (D1) and (D2).

Set $g(z_0,z_2) = (z_0+z_2)^l f(z_0+z_2,z_2) z_0^{-l}$. Since $z^{l+\frac{r}{T}}Y_M(u,z)w \in M[[z]]$, $z_1^l f(z_1, z_2) \in \mathbb{C}[[z_1,z_2,z_2^{-1}]]$, we obtain $g(z_0,z_2) \in \mathbb{C}((z_0,z_2))$. Then,

$$
\begin{aligned}
&\left\langle w', (z_0+z_2)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u,z_0+z_2) Y_M(v,z_2) w\right\rangle \\
&= \left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u,z_1) Y_M(v,z_2) w\right\rangle\Big|_{z_1=z_0+z_2} \\
&= f(z_1,z_2)(z_1-z_2)^{-l}\Big|_{z_1=z_0+z_2} = f(z_0+z_2,z_2) z_0^{-l} \\
&= (z_0+z_2)^l f(z_0+z_2,z_2) z_0^{-l} (z_0+z_2)^{-l} = g(z_0,z_2)(z_0+z_2)^{-l},
\end{aligned}
$$

that is, (D4). Now by (2.2),

$$
\begin{aligned}
&(z_0+z_2)^l \left\langle w', (z_0+z_2)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u, z_0+z_2) Y_M(v, z_2) w \right\rangle \\
&= (z_2+z_0)^l \left\langle w', (z_2+z_0)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(Y(u, z_0) v, z_2) w \right\rangle = g(z_0, z_2),
\end{aligned}
$$

from which one can deduce (D3) and (D5). □

Taking $v = \mathbf{1}$ in (2.2) we have

$$
(z_2+z_0)^{k+\frac{r}{T}} Y_M\left(e^{z_0 D} u, z_2\right) w = (z_0+z_2)^{k+\frac{r}{T}} Y_M(u, z_0+z_2) w, \tag{7.2}
$$

where D is a linear operator on V defined by $D(v) = v_{-2}\mathbf{1}$ for $v \in V$. Note that we can choose l large enough such that $z^{l+\frac{r}{T}} Y_M(u, z) w \in M[[z]]$. It follows that (7.2) can also be written as

$$
(z_2+z_0)^{l+\frac{r}{T}} Y_M\left(e^{z_0 D} u, z_2\right) w = (z_2+z_0)^{l+\frac{r}{T}} Y_M(u, z_2+z_0) w.
$$

Multiplying both sides by $(z_2+z_0)^{-l-\frac{r}{T}}$ gives

$$
Y_M\left(e^{z_0 D} u, z_2\right) w = Y_M(u, z_2+z_0) w. \tag{7.3}
$$

Now we are ready to present the proof of Theorem 2.1.

Proof. By Proposition 7.1, it is sufficient to deduce from the weak associativity that there exist $f(z_1, z_2) \in \mathbb{C}((z_1, z_2))$ and $g(z_0, z_2) \in \mathbb{C}((z_0, z_2))$ such that (D1)–(D5) hold. By (2.2) we have

$$
\begin{aligned}
&(z_0+z_2)^l \left\langle w', (z_0+z_2)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u, z_0+z_2) Y_M(v, z_2) w \right\rangle \\
&= (z_2+z_0)^l \left\langle w', (z_2+z_0)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(Y(u, z_0) v, z_2) w \right\rangle,
\end{aligned}
$$

which is denoted by $g(z_0, z_2)$. Clearly, $g(z_0, z_2) \in \mathbb{C}((z_0, z_2))$. Then it is easy to see that (D3) and (D4) hold. Choose l to be sufficiently large such that $z_0^l g(z_0, z_2) \in \mathbb{C}[[z_0, z_2, z_2^{-1}]]$. Set $f(z_1, z_2) = (z_1 - z_2)^l g(z_1 - z_2, z_2) z_1^{-l}$, which lies in $\mathbb{C}((z_1, z_2))$. Then, $f(z_2+z_0, z_2) z_0^{-l} = g(z_0, z_2)(z_2+z_0)^{-l}$, proving (D5); and

$$
\begin{aligned}
&\left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u, z_1) Y_M(v, z_2) w \right\rangle \\
&= \left\langle w', (z_0+z_2)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(u, z_0+z_2) Y_M(v, z_2) w \right\rangle \Big|_{z_0 = z_1 - z_2} \\
&= g(z_0, z_2)(z_0+z_2)^{-l} \Big|_{z_0 = z_1 - z_2} = g(z_1 - z_2, z_2) z_1^{-l} = f(z_1, z_2)(z_1 - z_2)^{-l},
\end{aligned}
$$

proving (D1).

By (D1), there exists $F(z_1,z_2) \in \mathbb{C}((z_1,z_2))$ such that $\left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(v,z_2) Y_M(u, z_1)w \right\rangle = F(z_1,z_2)(-z_2+z_1)^{-l}$ and $z_2^l F(z_1,z_2) \in \mathbb{C}[[z_1, z_1^{-1}, z_2]]$. Then,

$$
\begin{aligned}
&\left\langle w', z_1^{\frac{r}{T}} (-z_0+z_1)^{\frac{s}{T}} Y_M(v,-z_0+z_1) Y_M(u,z_1) w \right\rangle \\
&= \left\langle w', z_1^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(v,z_2) Y_M(u,z_1) w \right\rangle \Big|_{z_2=-z_0+z_1} \\
&= F(z_1,-z_0+z_1) z_0^{-l}.
\end{aligned}
$$

Thus,

$$
\left\langle w', z_1^{\frac{r}{T}} (z_1-z_0)^{\frac{s}{T}} Y_M(Y(v,-z_0)u,z_1) w \right\rangle = (-z_0+z_1)^l F(z_1,-z_0+z_1) z_0^{-l} (z_1-z_0)^{-l}
$$

by the weak associativity. Then,

$$
\begin{aligned}
f(z_2+z_0,z_2) z_0^{-l} &= g(z_0,z_2)(z_2+z_0)^{-l} \\
&= \left\langle w', (z_2+z_0)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M(Y(u,z_0)v,z_2) w \right\rangle && \text{(by (D3))} \\
&= \left\langle w', (z_2+z_0)^{\frac{r}{T}} z_2^{\frac{s}{T}} Y_M\left(e^{z_0 D} Y(v,-z_0)u,z_2\right) w \right\rangle \\
&= \left\langle w', z_2^{\frac{s}{T}} (z_2+z_0)^{\frac{r}{T}} Y_M(Y(v,-z_0)u,z_2+z_0) w \right\rangle && \text{(by (7.3))} \\
&= \left\langle w', z_1^{\frac{r}{T}} (z_1-z_0)^{\frac{s}{T}} Y_M(Y(v,-z_0)u,z_1) w \right\rangle \Big|_{z_1=z_2+z_0} \\
&= \left((-z_0+z_1)^l F(z_1,-z_0+z_1) z_0^{-l}\right)(z_1-z_0)^{-l} \Big|_{z_1=z_2+z_0} \\
&= F(z_2+z_0,z_2) z_0^{-l}.
\end{aligned}
$$

Thus, $F(z_2+z_0,z_2) = f(z_2+z_0,z_2)$ and then $F(z_1,z_2) = f(z_1,z_2)$, that is, (D2). This completes the proof. □

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Data availability

No data was used for the research described in the article.

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